Ambiguous Leverage Cycles[*]

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Abstract

Financial crises often originate in debt markets, where collateral constraints and opacity of asset values are the norm. In such ambiguous contexts endogenous beliefs formation is essential in explaining dynamics. We introduce *ambiguity attitudes*, which include both ambiguity aversion and seeking and which endogenously induce *optimism* (right-skewed beliefs) in booms and *pessimism* (left-skewed beliefs) in recessions, in a model where borrowers face occasionally binding collateral constraints. We use GMM estimation with latent value functions to estimate the ambiguity attitudes process. By simulating a crisis scenario in our model we show that optimism in booms is responsible for higher asset price and leverage growth and pessimism in recessions is responsible for sharper de-leveraging and asset price bursts. Analytically and numerically (using global methods) we show that our ambiguity attitudes coupled with the collateral constraints help to explain the frequency of crises, as well as asset price and debt cycle facts. At last, introducing an *intermediation channel* (credit supply) we show that it contributes to the severity of the crisis and to debt pro-cyclicality, while preserving the role of ambiguity.

JEL: E0, E5, G01

Keywords: *ambiguity attitudes, occasionally binding constraints, kinked multiplier preferences, leverage cycle, asset price cycle*

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1   Introduction

Most financial crises originate in debt markets and asset price as well as leverage cycles have important effects on the real economy. Opacity and collateral constraints are the two most notable features of debt markets and both can be a source of instability (See Holmstrom (2015)). First, collateral constraints expose debt markets to the fluctuations in collateral values and the anticipatory effects associated to their endogenous changes trigger large reversal in debt and asset positions. Second, agents trading in debt markets hold doubts about the fundamental value of the collateral. In this context ambiguity attitudes and endogenous beliefs formation are crucial in determining the dynamic of asset values and debt, also since the latter is tied to the first through the collateral constraint. The surge in asset prices and leverage observed prior to most financial crises and their collapse observed following it have often been linked to a combination of institutional factors, captured by collateral constraints, and endogenous beliefs formation. Optimism in booms, generated by assigning higher subjective beliefs to gains than to losses, can explain the surge in asset demand, prices and, through the collateral channel, in debt. Pessimism in recessions produces the opposite chain of events[1]. Despite the joint relevance of those elements in explaining the unfolding of financial crises, as well as the dynamic of asset prices and leverage over the business cycle, they are absent from the literature.

We fill this gap by assessing the role of ambiguity attitudes in a small open economy model where borrowers, investing in risky assets, are subject to occasionally binding collateral constraints that tie the scarcity or availability of debt to asset valuations. The latter is then affected by ambiguity attitudes, which render beliefs formation endogenous. Indeed the borrower, endowed with a sequence of subjective beliefs upon which he holds different amount of confidence, optimally chooses the degree of entropy, namely the distance between subjective and objective probability distributions, subject to bounds on it. The confidence in subjective beliefs are captured by an ambiguity parameter. Given the optimal entropy or likelihood ratio (LR hereafter), which affects also the value of risky assets through the stochastic discount factor (SDF here-after), the borrower solves optimal portfolio and leverage decisions.

Importantly we depart from the standard ambiguity aversion framework[2] and consider preferences which combine ambiguity aversion and ambiguity seeking. We model a dynamic generalization of the biseparable preferences axiomatized in Ghirardato and Marinacci (2001) and Ghirardato, Maccheroni and Marinacci (2004), which convexity the decision maker problem of finding the optimal beliefs by combining (depending on the weights) both aversion and seeking behaviour. Extended ambiguity attitudes have also strong support in experimental studies[3]. Specifically we model the decision marker problem using dynamic Lagrangian preferences a' la Hansen and Sargent (2001) and we then mix them to combine the entropy minimization problem (ambiguity aversion) and maximization problem (ambiguity seeking). Consistently with Ghi-

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[1]See Barberis (2011) for discussion on the role of over-confidence and under-confidence in particular for asset prices and leverage also at around the 2007-2008 financial crisis.

[2]See pioneering work by Hansen and Sargent (2001, 2007b) and Maccheroni, Marinacci and Rustichini (2006).

[3]Ambiguity seeking is strongly supported in experimental evidence. See Dimmock et al. (2015), Dimmock et al. (2016), Baillon et al. (2017) and Trautmann and van de Kuilen (2015) among others.

rardato, Maccheroni and Marinacci (2004) the weight or the indicator function in the optimal decision problem depends upon expected utilities. To validate our preferences empirically we determine the mapping between the ambiguity attitudes and the expected utility through structural estimation of the model. Specifically, we develop a novel estimation method by adapting the non-linear method of moments to our model-based combined Euler equation, in debt and risky asset[4]. We find that ambiguity aversion prevails when the value function is below its expected value (a case which we often label the loss domain) and viceversa. Those attitudes endogenously result in optimism or right-skewed beliefs in booms and pessimism in recessions[5]. This structure of the beliefs coupled with the anticipatory effects, which are typically associated with occasionally binding collateral constraints[6], have important implications for asset price, debt capacity and leverage dynamic. Consider a boom. Borrowers endogenously tend to act optimistically and increase their demand of risky assets. This boosts asset prices and through anticipatory effects also the demand of debt, which in turn endogenously relaxes the constraint. This is also consistent with the fact that in booms the evaluation of optimistic agents drives the debt capacity. The opposite is true in the loss domain. Ambiguity aversion typically induces persistence, but little volatility. Our preferences which combine the two in a kinked fashion induce the right amount of persistence and volatility needed to match asset price facts and debt dynamic.

With the above model we obtain a series of analytical and numerical results related to asset prices and debt dynamic. Analytically we discuss implications for asset prices and the Sharpe ratio. For the first, we show that the conditional LR heightens asset price growth in booms and depresses it in recessions. Second, the kink in the stochastic discount factor induced by the shift from optimism to pessimism helps to move the model-based Sharpe ratio closer to the Hansen and Hansen and Jagannathan (1991) bounds.

Next, we solve our model numerically by employing global non-linear methods with occasionally binding constraints[7]. The policy functions and a simulated crisis event, which allow us to discuss the economic intuition behind our model, show that optimism increases the build-up of leverage in booms, while pessimism steepens the recessionary consequence of the crisis. In both cases the comparison is done relatively to a model featuring solely collateral constraints, but no deviations between subjective and objective beliefs. Ambiguity attitudes play a crucial role in this result. In booms optimism boosts collateral values, hence, by relaxing the constraint,

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[4]For this we use the procedure developed in Chen, Favilukis and Ludvigson (2013), where one step involves the estimation of a latent unobservable variable given by the continuation value ratio.

[5]Our macro estimates are well in line with experimental evidence. Abdellaoui et al. (2011) provide foundations for S-shaped preferences with changing ambiguity attitudes and show through experimental evidence that pessimism (left-skewed beliefs) prevails in face of losses, while optimism prevails in face of gains. Further experimental evidence by Boiney (1993) Kraus and Litzenberger (1976) has associated ambiguity seeking (aversion) with right (left) skewed beliefs. On another front, survey evidence by Rozsypal and Schlafmann (2017), shows that low-income households hold pessimistic beliefs about the future, while the opposite is true for high-income households.

[6]Mendoza (2010) shows that the occasionally binding nature of the collateral constraints gives a role to anticipatory effects. As agents expect the constraint to bind in the future, they off-loads risky assets and debt in anticipation.

[7]We employ policy function iterations based on a Tauchen and Hussey (1991) discretization of the state space and by accommodating different regimes (portions of the state space) with binding or non-binding constraints.

it facilitates the build-up of leverage, asset demand and the asset price boom. In recessions pessimism materializes, which drives the transmission channel in the opposite direction. To subject our model and belief formations process to further rounds of empirical validation, we calibrate all parameters by minimizing the distance between some targeted model-based moments and their empirical counterparts using data for the US economy over the sample 1980-2016, namely the sample of both a rapid growth in leverage and then a sudden collapse in debt positions. Under the optimized calibration, the model can match asset price volatilities and equity premia (both the long run and the dynamic pattern), returns, Sharpe ratios, volatilities of debt and its pro-cyclicality [8]. The comparison with the model featuring solely the collateral constraint shows that our model performs better in the data matching. To explain asset price facts borrowers' ambiguity attitudes over the tails are crucial.

The rest of the chapter is structured as follows. Section 2 compares the paper to the literature. Section 3 describes the model and the ambiguity attitudes specification. Section 4 presents the estimation procedure and results. Section 5 investigates analytical results. Section 6 discusses quantitative findings (the solution method is detailed in the appendix). Section 7 concludes.

2   Comparison with Past Literature

Following the 2007 financial crisis which was triggered by panics in various debt markets (for structured products, for short-term bank funding and in repo markets, see Gorton and Metrick (2012)) there has been a growing interest in understanding the determinants and the dynamics of the leverage cycle and the role of the underlying externalities (pecuniary and demand) for the real economy. Most recent literature tends to assess the dynamic of debt over the business cycle through models with occasionally binding constraints. Papers on this topic include Geanakoplos (2010), Lorenzoni (2008), Mendoza (2010), which among many others examine both positive and normative issues related to the leverage cycle. Papers focusing on the positive aspects show that anticipatory effects produced by occasionally binding constraints are crucial in generating sharp reversals in debt markets and in establishing the link between the tightening of the constraint and the unfolding of financial crisis. None of the past papers however assesses the joint role of financial frictions, in the form of collateral constraints, and belief formation, while both play a crucial role in determining the asset price and leverage cycle in normal times and in explaining endogenously the unfolding of crises even in face of small shocks. One exception is Boz and Mendoza (2014) which introduces learning on asset valuation in a model with occasionally binding collateral constraints. Contrary to them our beliefs are endogenously formed based on ambiguity attitudes toward model mis-specification. Moreover none of the past papers conducts a quantitative analysis aimed at assisting the quantitative relevance of those elements in jointly matching asset price and debt facts and cyclical moments.

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[8]It is well documented by Jorda, Schularick and Taylor (2016) at aggregate level and using historical data. But it is also well document for consumer debt, see for instance Fieldhouse, Livshits and MacGee (2016) among others.

The relevance of ambiguity and of the beliefs formation process is crucial in debt markets in which opacity is the norm (see Holmstrom (2015)). Indeed, contrary to equity markets in which buyers of the asset wish to exert monitoring and control on the investment activity, participants in debt markets usually trade under the ignorance of the fundamental value of collateral. For this reason in debt markets a collateral guarantee is part of the contractible set-up. This indeed serves the purpose of overcoming the pervasive asymmetric information. However even if the information asymmetry underlying the specific debt relation is solved through the contracts, doubts remain about the fundamental value of the asset, implying that optimism or pessimism of subjective beliefs affect the agents' saving and investment problem, hence the dynamic of asset prices and leverage. Despite the realism and importance of the connection between ambiguity and debt dynamic, this nexus has not been studied so far.

Since we choose to model endogenous beliefs formation through ambiguity attitudes our model is also connected to the literature on ambiguity aversion (see Hansen and Sargent (2001, 2007b) and Maccheroni, Marinacci and Rustichini (2006)). In this context some papers assess the role of ambiguity aversion for asset prices or for portfolio allocation. For instance Barillas, Hansen and Sargent (2007) show that ambiguity aversion is akin to risk-sensitive preferences a' la Tallarini (2000) and as such it helps the model's Sharpe ratio to get closer to the Hansen and Jagannathan (1991)[9]. Most of the papers focusing on ambiguity aversion are able to explain well price patterns persistence, but less so price volatility. To improve on the latter some papers introduce time-varying ambiguity aversion to study asset prices properties, such as Epstein and Schneider (2008), Drechsler (2013), Leippold, Trojani and Vanini (2008) and Bianchi, Ilut and Schneider (2017). Ilut and Schneider (2014), in particular, explain crises with a loss of confidence obtained by adding a shock to the ambiguity averse framework. All of the above papers focus on ambiguity aversion, which induces pessimism, none introduces ambiguity seeking, which induces optimism.

We depart from this literature in two important ways. First, we model ambiguity attitudes that encompass both ambiguity aversion and ambiguity seeking behavior. Our preferences are indeed a dynamic generalization of the biseparable preferences axiomatized in a static context by Ghirardato and Marinacci (2001) and Ghirardato, Maccheroni and Marinacci (2004). Both papers show that ambiguity attitudes can be formalized within a general decision model by constructing a biseparable preference, which can combine both ambiguity aversion and ambiguity seeking. Effectively preferences are mixed with respect to the problem of finding the optimal beliefs, so that under a weight of one the decision maker solves a minimization problem (ambiguity aversion) and viceversa. The weights in their formalization depend upon expected utility mapping. In our work we construct a value function, which embeds a multiplier on the entropy, that can be mixed, thereby combining ambiguity aversion (with a positive multiplier on entropy) and ambiguity seeking (negative multiplier). Consistently with Ghirardato, Maccheroni and Marinacci (2004), the indicator function, which non-linearly shifts the preferences from pessimistic to its dual, depends upon the deviations of the current value function from

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[9]On a different line of research Benigno and Nisticó (2012) show how ambiguity averse preferences can be used to explain the home bias in international portfolio allocations due to the need to hedge against long run risk.

a reference level, represented by its mean. Importantly, our state-contingent multiplier preferences are estimated. This not only validates empirically the preferences, but it also allows us to pin down the exact form of the state contingency in the multiplier (negative in the gain domain and positive in the loss domain). Equipped with these preferences, we show that beliefs endogenously become pessimistic in the loss domain (when the value function is below its mean) and optimistic in the gain domain (the opposite case). This has important consequences in our case. Indeed, by embedding those preferences into a leverage cycle and risky investment problem we can show that optimism induces price acceleration and excessive leverage, while pessimism induces the opposite. Moreover, the combination of ambiguity attitudes with an occasionally binding collateral constraint delivers the right amount of persistence and volatility needed to explain jointly asset price and debt dynamic. At last the kinked nature of the preferences helps in generating the right volatility in the Sharpe ratios, which governs risk-taking behaviour.

Note that ambiguity seeking as well as the state contingent nature of the ambiguity attitudes is also well documented in experimental studies (see Dimmock et al. (2015), Baillon et al. (2017), Trautmann and van de Kuilen (2015), and Roca, Hogarth and Maule (2006) among others). We confirm the existence and significance of ambiguity attitudes through time-series estimation or our model. Furthermore, it is only by accounting jointly for ambiguity aversion and ambiguity seeking that our model is able to match numerically the volatilities, the persistence and the cyclical behavior of asset prices and debt.

At last, our paper relates to the extensive literature on the estimation of the SDF with behavioural elements. More closely, we build upon the latent factor estimation method of Chen, Favilukis and Ludvigson (2013). We depart from them along the following dimensions. First, we adapt their estimation procedure to preferences with state-contingent ambiguity attitudes and in presence of an occasionally binding collateral constraint. Secondly, our latent factor is derived analytically, while in their case it is estimated semi-nonparametrically.

3   A Model of Ambiguous Leverage Cycle

Our baseline model economy is an otherwise standard framework with borrowers facing occasionally binding collateral constraints. One of the novel ingredients stems from the interaction between ambiguity attitudes and debt capacity. Debt supply is fully elastic with an exogenous debt rate as normally employed in most recent literature on the leverage cycle.[10] Collateral in this economy is provided by the value of the risky asset funded through debt. To this framework we add ambiguity attitudes, which includes both ambiguity aversion and seeking. The underlying logic is similar to the one pioneered and proposed by the game-theoretic approach a? la Hansen and Sargent (2007a) in which agents are assumed to have fears of model mis-specification and play a two-stage game with a malevolent agent (nature) that amplifies deviations from the true probability model and helps the borrower to explore the fragility of a decision rule with respect to various perturbations of the objective shock distribution. Below, we show that am-

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[10]This model economy corresponds to a limiting case in which lenders are risk-neutral. Alternatively the model can be interpreted as a small open economy with debt supplied from the rest of the world.

biguity aversion results endogenously in left-skewed or pessimistic beliefs, relatively to rational expectation, namely relatively to the case in which objective and subjective beliefs coincide. On the other side ambiguity seeking results in right-skewed or optimistic beliefs. Importantly the changing nature of the ambiguity attitudes contributes to the occasionally binding nature of the collateral constraint. As agents become optimist their demand for risky assets contributes to boost collateral values and to expand debt capacity. The opposite is true with pessimism.

3.1 Beliefs Formation and Preferences

The source of uncertainty in the model is a shock to aggregate income y_t, which is our exogenous state and follows a finite-space stationary Markov process. We define the state space as S_t, the realization of the state at time t as s_t and its history as $s^t = \{s_0, s_1, \ldots, s_t\}$ with associated probability $\pi(s^t)$. The initial condition of the shock is known and defined with s_{-1}.

Borrowers are endowed with the approximating model $\pi(s^t)$ over the history s^t but they also consider alternative probability measures, indicated by $\tilde{\pi}(s^t)$, which deviate from $\pi(s^t)$.[11] Borrowers can have different degrees of trust in their own subjective beliefs, so that act as ambiguity averse when they fear deviations from the approximated model and they act as ambiguity seeking when they hold high confidence in their beliefs. Following the relevant literature, we introduce the measurable function $M(s^t) = \tilde{\pi}(s^t)/\pi(s^t)$, which we define as the likelihood ratio. We can also define the conditional likelihood ratio as, $m(s_{t+1}|s^t) = \tilde{\pi}(s_{t+1}|s^t)/\pi(s_{t+1}|s^t)$. For ease of notation since now onward we use the following notation convention: $M_t = M(s^t), M_{t+1} = M(s^{t+1})$ and $m_{t+1} = m(s_{t+1}|s^t)$, where the sub-index refers to the next period state. The above definition of M_t allows us to represent the subjective expectation of a random variable x_t in terms of the approximating probability models:

$$\tilde{\mathbb{E}}_t\{x_t\} = \mathbb{E}_t\{M_t x_t\} \tag{1}$$

where $\mathbb{E}_t$ is the subjective expectation operator conditional to information at time t for the probability $\pi(s^t)$, while $\tilde{\mathbb{E}}_t$ is the expectation operator conditional to information at time t for the probability $\tilde{\pi}(s^t)$. The function M_t follows a martingale process and as such it satisfies the following condition $\mathbb{E}\{M_{t+1}\} = M_t$. We can decompose M_t as follows

$$m_{t+1} \equiv \frac{M_{t+1}}{M_t} \qquad \text{for} \quad M_t > 0 \tag{2}$$

and $m_{t+1} = 1$ for $M_t = 0$. These incremental deviations satisfy condition $\mathbb{E}_t\{m_{t+1}\} = 1$. Moreover, the discrepancy between the approximating and the subjective models is measured by the conditional entropy, defined as follows:

$$\varepsilon(m_{t+1}) = \mathbb{E}_t\{m_{t+1}\log m_{t+1}\} \tag{3}$$

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[11] The alternative probability measure $\tilde{\pi}$ is absolutely continuous with respect π. This means that events that receive positive probability under the alternative model, also receive positive probability under the approximating model

where $\varepsilon(m_{t+1})$ is a positive-valued, convex function of $\pi(s^t)$ and is uniquely minimized when $m_{t+1} = 1$, which is the condition characterizing the case with no ambiguity attitudes. Given the above probabilistic specification, we characterize the *kinked multiplier preferences* as follows:

$$
V(c_t) = \begin{cases} \min_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^+ \varepsilon(m_{t+1}) \right\} & \text{if} \quad V_t \leq \mathbb{E}_{t-1}\{V_t\} \\[2em] \max_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^- \varepsilon(m_{t+1}) \right\} & \text{if} \quad V_t > \mathbb{E}_{t-1}\{V_t\} \end{cases}
\tag{4}
$$

where $u(c_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$. The kink in preferences is related to a threshold defined in terms of the difference between the agent's value function V_t and its historical mean $\mathbb{E}V_t$. The latter is interpreted as a proxy of the current state of the economy. When the condition holds with equality ($V_t = \mathbb{E}_{t-1}\{V_t\}$), this preference structure collapses to the rational expectation representation, namely $V_t = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$. In the above expression, $\theta \in \mathbb{R}$ is a process capturing the degree of doubts about the prevailing model, and it is a state-contingent binary variable which takes positive (negative) values for states of the world for which the value function is below (above) its average. Mathematically the value function under θ^- is essentially the dual representation of the value function under θ^+. The state-contingency in preferences implies that ambiguity aversion (thus pessimism) prevails in bad states, while ambiguity seeking (thus optimism) prevails in good states. This property will be object of empirical validation in Section 4.

3.2 Preferences Formalization

The kinked multiplier preferences can be derived as a dynamic generalization of the biseparable preferences axiomatized in Ghirardato and Marinacci (2001) and Ghirardato, Maccheroni and Marinacci (2004). Both papers show that ambiguity attitudes can be formalized within a general decision model by constructing a biseparable preference, which can combine both ambiguity aversion and ambiguity seeking. Preferences are mixed with respect to the problem of finding the optimal belief. Consider the instantaneous utility function, $u(c_t)$, and the problem of finding the optimal belief. Given again the specification, in equation 4, we can represent our kinked multiplier preferences as follows:

$$
V_t(c_t) = \mathbb{I}_{\theta_t \geq 0} \min_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta \varepsilon(m_{t+1}) \right\} +
$$
$$
\mathbb{I}_{\theta < 0} \max_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta \varepsilon(m_{t+1}) \right\}
\tag{5}
$$

As noted in Ghirardato, Maccheroni and Marinacci (2004) the indicator function shall depend only upon expected utility mapping. We design the following expected utility mapping so that $\theta_t < 0$ whenever $V_t > \mathbb{E}_{t-1}\{V_t\}$ (which since now we often refer as the gain domain) and

viceversa (in the loss domain). We can therefore re-write our preferences as:

$$V_t(c_t) = \mathbb{I}_{V_t \leq \mathbb{E}_{t-1}\{V_t\}} \min_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^+ \varepsilon(m_{t+1}) \right\} +$$

$$\mathbb{I}_{V_t > \mathbb{E}_{t-1}\{V_t\}} \max_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^- \varepsilon(m_{t+1}) \right\} \qquad (6)$$

Three theoretical notes are worth at this point. First, as noted in Ghirardato, Maccheroni and Marinacci (2004) most decision theory models of ambiguity employ those biseparable preferences, but add additional assumptions. For instance ambiguity aversion arises under the assumption of ambiguity hedging, namely the fact that between two indifferent alternatives the ambiguity averse decision marker prefers a convex combination of the two to each one in isolation. Under ambiguity seeking this assumption should be reversed. Second, the dependence of the indicator function upon the expected utility effectively creates a dependence with respect to the state of the economy. Indeed it is only after a sequence of negative shocks to wealth that the value function passes its mean and viceversa[12]. Therefore, formally we should condition the indicator function and the θ_t upon the state of the economy. With a slight abuse of notation and for convenience we maintain our notation of a time dependent θ_t as in the context of our model we deal with random shocks in a time series context. At last, note that the general formalization of the decision problem is not explicit about the exact dependence of the indicator function upon the gain or the loss domain. This is effectively an empirical question. Indeed as explained above, it has been addressed in the context of experimental studies. For this reason later below we estimate our model and we assign to θ_t a state-contingent process which is consistent with the data and the evidence that we find. This effectively also serves as an indirect validation of the preferences. Note that for robustness we run two types of estimation. The first is a reduced form through GMM confined to the model-implied Euler equation, the second is a method of moments on the entire model. Both methods give the same consistent answer, albeit understandably they deliver two different values for the estimated parameters.

Some additional considerations are worth on the interpretation of our preferences and on their implication for the asset price and the leverage cycle. First, as we show below, when solving the decision maker problem of finding the optimal beliefs, our kinked multiplier preferences deliver pessimism (or left-skewed) beliefs in loss domain and optimism (right-skewed) beliefs in gain domain. Framed in the context of the Hansen and Sargent (2007a) game with nature, the optimal belief problem has the following interpretation. Under the loss domain nature acts malevolently and increases uncertainty. In this way nature also tests the decision maker, who fearing uncertainty acts pessimistically, hence assigns more weights to adverse states. In a consumption-saving problem this naturally induces more precautionary saving, while in our framework, where financial crises endogenously materialize, pessimistic beliefs are responsible for stronger deleveraging (and fire sales) during the downturn. This effect is well in line with

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[12]In this respect the preferences are also akin to the news dependent preferences a' la Kőszegi and Rabin (2007). See also recently Pagel (2014). The main difference is that news dependence affects risk aversion, while in our case it affects attitudes toward uncertainty. Second, once again we consider aversion but also its dual.

post-crises dynamic. Under the gain domain nature acts benevolently and reduces uncertainty. This induces the decision maker to take more risk and assign more weight to the upper tail[13]. This leads to the emergence of risk-taking and excessive leverage. In both cases nature shifts decision makers? behaviour toward the tails. Hence, our preferences are well in line with the prevalent interpretation of model ambiguity. As we show extensively below however considering ambiguity seeking and extended attitudes helps greatly in explaining asset price facts as well as in the context of our leverage model also debt dynamic.

3.3 Budget and Collateral Constraint

The rest of the model follows a standard leverage cycle model with risky assets that serve as collateral (see e.g. Mendoza (2010)). The representative agent holds an infinitely lived asset x_t, which pays a stochastic dividend d_t every period and is available in fixed unit supply. The asset can be traded across borrowers at the price q_t. In order to reduce the dimension of the state space, we assume that the dividend is a fraction α of the income realization. Therefore, we indicate with $(1-\alpha)y_t$ the labor income and with $d_t = \alpha y_t$ the financial income. Agents can borrow using one-period non-state-contingent bonds that pay an exogenous real interest rate R. The budget constraint of the representative agents can be expressed as following:

$$c_t + q_t x_t + \frac{b_t}{R} = (1-\alpha)y_t + x_{t-1}[q_t + d_t] + b_{t-1} \tag{7}$$

where c_t indicates consumption and b_t the bond holdings. The agents' ability to borrow is restricted to a fraction ϕ of the value of asset holding:

$$-\frac{b_t}{R} \leq \phi q_t x_t \tag{8}$$

The collateral constraint depends on the current period price of the asset in order to reproduce fire-sales driven amplification dynamics, which for this simple model would not be produced with a different formulation of the constraint.[14]

3.4 Recursive Formulation

Following Hansen and Sargent (2007b), we rely on the recursive formulation of the problem, which allows us to re-write everything only in terms of m_{t+1}. The recursive formulation shall of course be adapted to capture the changing nature of the ambiguity attitudes.

We now partition the state space S_t in the two blocks, given by the endogenous and the exogenous states, $S_t = \{B_t, y_t\}$, where B_t is the aggregate bond holdings and y_t the income realization. Note that the aggregate asset holdings is not a state variable because it is in fixed supply. Moreover, the problem is also characterized by the two individual state variables (b_t, x_t).

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[13]Given this interpretation, such beliefs formation process is also akin to the one considered in Brunnermeier and Parker (2005) in which a small optimistic bias in beliefs typically leads to first-order gains in anticipatory utility.

[14]Moreover, Bianchi and Mendoza (2015) provide a micro-founded derivation of this constraint, based on a limited enforcement problem.

For the recursive formulation we employ a prime and sub-index to indicate variables at time $t+1$ and no index for variables at time t. The borrowers' recursive optimization problem reads as follows. Conditional on $\theta_t > 0$, the recursive two-stage optimization reads as follows:

$$V(b, x, S) = \max_{c, x', b'} \min_{m'} \Big\{ u(c) + \beta \mathbb{E}_S \{ m'V(b', x', S) + \theta m' \log m' \} \tag{9}$$
$$+ \lambda \left[y + q(S)(x + \alpha y) + b - q(S)x' - c - \frac{b'}{R} \right]$$
$$+ \mu \left[\phi q(S)x' + \frac{b'}{R} \right] + \beta \theta \psi \left[1 - \mathbb{E}_S m' \right] \Big\}$$

Conditional on $\theta_t < 0$, the recursive two-stage optimization reads as follows:

$$V(b, x, S) = \max_{c, x', b'} \max_{m'} \Big\{ u(c) + \beta \mathbb{E}_S \left[m'V(b', x', S) - \theta m' \log m' \right] \tag{10}$$
$$+ \lambda \left[y + q(S)(x + \alpha y) + b - q(S)x' - c - \frac{b'}{R} \right]$$
$$+ \mu \left[\phi q(S)x' + \frac{b'}{R} \right] + \beta \theta \psi \left[1 - \mathbb{E}_S m' \right] \Big\}$$

where the aggregate states follow the law of motion $S' = \Gamma(S)$. In the above problem λ and μ are the multipliers associated to the budget and collateral constraints respectively, while the term $\beta\theta\psi$ is the multiplier attached to the constraint $\mathbb{E}_S[m'] = 1$.

The above optimization problems are solved sequentially. First an inner optimization and then an outer optimization problem are derived sequentially. In the first stage agents choose the optimal incremental probability distortion for given saving and portfolio choices. In the second stage, for given optimal likelihood ratio, they solve the consumption/saving problem and choose the optimal amount of leverage. Intuitively, the problem is modelled as a game of strategic interactions between the maximizing agents, who face Knightian uncertainty[15], and a malevolent/benevolent agent that draws the distribution (see Hansen and Sargent (2007b) who proposed this reading).

3.4.1 The Inner Problem

Through the inner optimization problem the borrowers choose the optimal entropy or conditional likelihood ratio, namely the optimal deviation between his own subjective beliefs and the objective probability distribution. The first order condition with respect to m', which is functionally equivalent under the two cases, is given by:

$$V(b', x', S') + \theta(\log m' + 1) - \theta\psi = 0 \tag{11}$$

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[15]Knight (1921) advanced the distinction between risk, namely the known probability of tail events, and uncertainty, namely the case in which such probabilities are not known. Ambiguity usually refers to cases of uncertainty where the state space is well defined, but objective probabilities are not available.

Rearranging terms, we obtain:

$$1 + \log m' = -\frac{V(b', x', S')}{\theta} + \psi$$

$$m' = \exp\left\{\frac{-V(b', x', S')}{\theta}\right\} \exp\{\psi - 1\} \tag{12}$$

Finally, imposing the constraint over probability deviation m', and defining $\sigma = -\frac{1}{\theta}$ we derive the optimality condition for the conditional likelihood ratio:

$$m' = \frac{\exp\{\sigma V(b', x', S')\}}{\mathbb{E}\left[\exp\{\sigma V(b', x', S')\}\right]} \tag{13}$$

Equation (13) also defines the state-contingent incremental probability deviation from the rational expectation case. The magnitude and the direction of this deviation depends on the agents' value function and the value for the inverse of σ. We will return on the role of the optimal conditional likelihood ratio later on.

3.4.2 The Outer Problem

For given optimal LR m' the borrower solves an outer optimization problem in consumption, risky assets and debt. Upon substituting the optimal LR into the value function, the maximization problem reduces to find the optimal allocations of consumption, bond holding and asset holdings. The resulting recursive problem is:

$$V(b, x, S) = \max_{c, x', b'}\left\{u(c) + \frac{\beta}{\sigma}\log\left[\mathbb{E}_S \exp\{\sigma V(b', x', S')\}\right]\right.$$

$$+ \lambda\left[y + q(S)((x + d) + b - q(S)x' - c - \frac{b'}{R}\right]$$

$$+ \mu\left[\phi q(S)x + \frac{b'}{R}\right]\right\} \tag{14}$$

We will now derive and list all the competitive equilibrium conditions. Since now we return to the notation with t and $t + 1$ indices as this is needed for our analytical derivations in section 5. The borrowers' first order condition with respect to bond holding and risky assets reads as follows:

$$u_c(c_t) = \beta R \mathbb{E}_t\{m_{t+1}u_c(c_{t+1})\} + \mu_t \tag{15}$$

$$q_t = \beta\frac{\mathbb{E}_t\{m_{t+1}u_c(c_{t+1})[q_{t+1} + \alpha y_{t+1}]\}}{u_c(c_t) - \phi\mu_t} \tag{16}$$

where u_c indicates the marginal utility of consumption. Equation (15) is the Euler equation for bonds and displays the typical feature of models with occasionally binding collateral constraint. In particular, when the constraint binds there is a wedge between the current and the expected future consumption marginal utility, given by the shadow value of relaxing the collateral constraint. Equation (16) is the asset price condition.

Note that ambiguity attitudes, hence beliefs, affect asset prices since m_{t+1} enters the optimality conditions for risky assets, equation (16), and they affect the tightness of the debt limit as m_{t+1} enters the Euler equation 15. In other words the optimal m_{t+1} affects the stochastic discount factor and through this it affects the pricing of all assets in the economy. The model characterization is completed with the complementarity slackness condition associated to the collateral constraint:

$$\mu_t \left[\frac{b_{t+1}}{R} + \phi q_t \right] = 0 \tag{17}$$

and with the goods and stock markets clearing conditions:

$$c_t + \frac{b_{t+1}}{R} = y_t + b_t \tag{18}$$

$$x_t = 1 \tag{19}$$

Definition 3.1 (Recursive Competitive Equilibrium). *A Recursive Competitive Equilibrium is given by a value function V_t, allocations $(c_t; b_{t+1})$, probability distortions m_{t+1} and prices q_t such that:*

- *given prices and allocations the probability distortions solve the inner problem;*

- *given prices and probability distortions, allocations and the value function solve the outer problem;*

- *the allocations are feasible, satisfying (18) and (19);*

- *the aggregate states' law of motion is consistent with agents' optimization;*

3.5   Pessimism and Optimism

To determine under which states the Lagrange multiplier, θ_t, turns positive or negative we will estimate our model implied Euler equations through GMM in the next section. In the meantime it is useful to discuss how the ambiguity averse or ambiguity seeking attitudes affect the endogenous formation of beliefs, as captured by the optimal likelihood ratio. For simplicity of exposition we report the optimal condition for variable m_{t+1}:

$$m_{t+1} = \frac{\exp\left\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\right\}}{\mathbb{E}_t\{\exp\left\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\right\}\}} \tag{20}$$

The conditional deviation affects how agents assign different subjective probabilities (with respect to the objective ones) to future events, which can be characterized by high and low utility. In particular, if $m_{t+1} > 1$ agents assign an higher subjective probability, while if $m_{t+1} < 1$ the opposite holds. Given this, the sign of the parameter σ_t affects how these conditions are linked to positive or negative future state realizations.[16] The following lemma summarizes this consideration and defines optimism and pessimism in the agents' attitude.

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[16]Concerning the size of the distortion, we can say that a large absolute value of θ increases the probability distortion in all future states, meaning that m' is close to unity. At the contrary, a small absolute value of θ, implies that the decisions are far from the rational expectation setting.

Lemma 3.2. *When $\theta_t < 0$ $m_{t+1} > 1$ in good states and $m_{t+1} < 1$ in bad states. Hence, beliefs endogenously emerge as right-skewed and agents act with optimism. When $\theta_t > 0$ the opposite is true.*

Proof. First we define good states as those in which the current state value function is above its expected value. When $\theta_t < 0$; then $\sigma_t > 0$ in good states $\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\} > \mathbb{E}_t\{\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\}\}$ namely the risk-adjusted value function for the good states is larger than the average one. Based on the above equation, this implies that $m_{t+1} > 1$. The opposite is true in bad states. When $\theta_t > 0$ then $\sigma_t < 0$ this implies that in good states $\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\} < \mathbb{E}_t\{\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\}\}$, namely the risk-adjusted value function for the good states is lower than the average one and $m_{t+1} < 1$. The opposite is true in bad states.

3.5.1 Beliefs Formation: A binomial state space example

To gain some intuition we discuss a particular case with only two income states, which we define as high, with a sup-index h, and low, with a sup-index l. We also consider only two periods which we label as $t = 0, 1$. By assumption the high state is high enough that the collateral constraint is slack, while the opposite is true for the low state. This facilitates the computation of the expectation operators. The states have a binomial probability structure such that state h realizes with probability π, while the state l with its complement $1 - \pi$. Equipped with these assumptions we can characterize the dynamic between time 0 and time 1. In this case the likelihood ratio can be specified as follows:

$$m_1 = \frac{\exp\{\sigma_0 V_1\}}{\pi \exp\{\sigma_0 V_1^h\} + (1-\pi) \exp\{\sigma_0 V_1^l\}} \tag{21}$$

where $V_1^h > \mathbb{E}_0\{V_1\}$ and $V_1^l < \mathbb{E}_0\{V_1\}$. Note that depending on the time zero realization of the state we have two different values of the inverse of the penalty parameter, σ_0. To fix ideas imagine that the income realization at time zero is the low state, l. Given our Lemma 3.2 we have that $\sigma_o^l < 0$. The latter implies that $\exp\{\sigma_0^l V_1^h\} < \mathbb{E}_0\{\exp\{\sigma_0^l V_1\}\}$ and $\exp\{\sigma_0^l V_1^l\} > \mathbb{E}_0\{\exp\{\sigma_0^l V_1\}\}$. Therefore, the marginal likelihood ratio are $m_1^h < 1$ and $m_1^l > 1$. As a consequence, we can define the following subjective probabilities as:

$$\omega^h = \pi m_1^h < \pi \quad \omega^l = (1-\pi)m_1^l > (1-\pi) \tag{22}$$

As we can see, agents assign a higher (lower) subjective probability - with respect to the objective probability - to the future negative (positive) events, typical of a pessimistic attitude. The opposite is true when $\sigma_o^l < 0$. In this case $\exp\{\sigma_0^h V_1^h\} > \mathbb{E}_0\{\exp\{\sigma_0^h V_1\}\}$ and $\exp\{\sigma_0^h V_1^l\} < \mathbb{E}_0\{\exp\{\sigma_0^h V_1\}\}$ producing $m_1^h > 1$ and $m_1^l < 1$.

Therefore, agents assign higher (lower) subjective probability to the future positive (negative) events, showing an optimism attitude:

$$\omega^h = \pi m_1^h > \pi \quad \omega^l = (1-\pi)m_1^l < (1-\pi) \tag{23}$$

The interesting feature of this state-contingent behaviour concerns its connections with asset prices, the value of collateral and leverage. Further below we explain this in more details through analytical derivations and quantitative analysis. Intuitively, optimism explains why asset price booms and leverage build-ups are steeper in booms and relatively to the model with no beliefs formation. To fix ideas consider the case with a negative θ_0 and that the borrower experiences a good state today and expects a good state tomorrow. Asset price would grow even in the case with no ambiguity attitudes, however under our extended multiplier preferences, borrowers form today subjective beliefs that induce an LR of $m_1^h > 1$. As this makes the borrowers' SDF right-skewed distributed, it induces higher demand for both. This is why we label this case as optimism.

Consider now the opposite case, namely θ_0 lower than zero. According to Lemma 3.2 now the optimal LR is left skewed, namely lower than one if associated to future good states and larger than one to bad states. In other words the borrower becomes pessimistic. In this case, if a bad state is expected asset prices will fall according to equation 16 and they would do so more sharply than under when $m_1 = 1$ across all states of nature. Hence we shall conclude that pessimism explains why asset price bursts and de-leverages are sharper in recessions and relatively to the case with no ambiguity attitudes. Appendix F considers a more extended version of the three periods model and also shows analytically that our ambiguity attitudes interacting with the collateral constraint induces higher debt levels in booms. Further below we explain through analytical derivations of the full dynamic model and through simulations of it how the ambiguity attitudes contribute to explain asset price and debt dynamics.

4   Estimation of the Model Implied SDF

To provide empirical ground to how ambiguity attitudes are formed depending on the current state of the economy we estimate the model-implied Euler equations. This delivers a process for θ_t, whose state-contingent nature empirically supports our preference specification.

We devise a novel estimation method apt to a model with collateral constraints and kinked multiplier preferences. It is based on adapting the minimum distance estimation conditional on latent variables to our modelling environment. In a nutshell we derive a moment condition by using the combined non-linear expression for the Euler equations (15) and (16). As we show in Appendix A, the latter depends on the value function. We therefore follow the approach in Chen, Favilukis and Ludvigson (2013), who write the Euler moment condition as function of the estimated value function. A crucial difference between our method and theirs is that their value function has an unknown functional form, which is estimated semi-nonparametrically, while ours can be derived analytically. Specifically, following Hansen, Heaton and Li (2008), we derive its functional form, which is then estimated using maximum likelihood.

More specifically, the estimation procedure (whose detailed derivations are contained in Appendix A) can be described as follows. First, one shall re-write the value function in terms of an ambiguity factor. For this, we adapt the steps used in the recursive preference literature to the case of our kinked multiplier preferences (see Appendix A.1). Next, the implied SDF is

derived (see Appendix A.2) and the value function is estimated (see Appendix A.3). Substitution of the derived SDF into the combined Euler equations for debt and risky assets, (15) and (16), delivers the final moment condition (see Appendix A.4). At last, as it is common for GMM estimation, we condition on a set of instruments, $\mathbf{z_t}$. The resulting moment condition reads as follows:

$$\mathbb{E}_t \left\{ \left[\underbrace{ \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1} \left(\frac{\frac{\exp(V_{t+1})}{c_{t+1}} \frac{c_{t+1}}{c_t}}{\sqrt[\beta]{\frac{\exp(V_t)}{c_t}}} \right)^{\sigma_t} }_{\Lambda_{t,t+1}} \left(R_{t+1}^s - \phi R_{t+1} \right) + \phi - 1 \right] \mathbf{z_t} \right\} = 0 \qquad (24)$$

where $R_{t+1}^s = \frac{q_{t+1} + d_{t+1}}{q_t}$ is the cum-dividend return on risky asset and R_{t+1} is the risk-free interest rate, which is time-varying in the data. Note that the expression for the SDF can be decomposed into two factors, $\Lambda_{t,t+1}^1 = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1}$ and $\Lambda_{t,t+1}^2 = \left(\frac{\frac{\exp(V_{t+1})}{c_{t+1}} \frac{c_{t+1}}{c_t}}{\sqrt[\beta]{\frac{\exp(V_t)}{c_t}}} \right)^{\sigma_t}$, where the second captures the role of ambiguity attitudes. Equation (24) is estimated fully non-linearly with GMM methods.[17] Note that tight restrictions are placed on asset returns and consumption data since our moment condition embodies both financial and ambiguity attitudes. For the estimation we fix the loan to value ratio at $\phi = 0.5$ and, given that $\theta_t = -\frac{1}{\sigma_t}$, we estimate the preference parameters, β and θ_t.

Regarding the data, we use real per capita expenditures on non-durables and services as a measure of aggregate consumption. For R we use the three-month T-bill rate, while R^s is proxied through the Standard & Poor 500 equity return.[18] The choice of the instruments follows the literature on time-series estimation of the Euler equations.[19] They are grouped into internal variables, namely consumption growth and interest rates two quarters lagged, and external variables, namely the value and size spreads, the long-short yield spread and the dividend-price ratio (see also Yogo (2006)). A constant is additionally included in order to restrict model errors to have zero mean. Finally, the model's over-identifying restrictions are tested through the J-test (test of over-identifying restrictions, Hansen (1982)). [20]

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[17]Optimal GMM parameters minimize a quadratic loss function over the weighted distance between population and sample moments, by a two-step GMM.

[18]Data sources are NIPA Tables https://www.bea.gov/iTable/index_nipa.cfm, CRSP Indices database http://www.crsp.com/products/research-products/crsp-historical-indexes, and the Shiller database http://www.econ.yale.edu/~shiller/data.htm, respectively

[19]See Stock, Wright and Yogo (2002) for a survey on the relevance of instruments choice in a GMM setting

[20]This is a specification test of the model itself and it verifies whether the moment conditions are enough close to zero at some level of statistical confidence, if the model is true and the population moment restrictions satisfied.

Table 1: Estimation Results

Sample	Estimated parameters				
	β	θ	$\theta(\tilde{v}_t > E\tilde{v}_t)$	$\theta(\tilde{v}_t \leq E\tilde{v}_t)$	$J - test$
1980-2016	0.982		-1.701	2.434	4.385
	(.022)		(.053)	(.075)	(.495)
1985:Q1-2007:Q2	0.891	-1.959			3.811
	(.058)	(.238)			(.702)
2007:Q3-2016:Q4	0.879	7.4404			2.026
	(.015)	(.022)			(.917)

In parenthesis: the HAC standard errors for the parameter estimates; the p-values for the J-test

Table 1 presents the results. The estimated values of θ_t are conditioned to the logarithm of the continuation value ratio, defined as $\tilde{v}_t = V_t - \log(c_t)$. Consistently with our previous definition, good states are those for which the latent value function is higher than its mean and vice-versa for bad states. Column 3 shows results conditioned upon the relation $\tilde{v}_t > \mathbb{E}\{\tilde{v}_t\}$, while column 4 reports the results for the complementary condition. We find that a negative value (-1.701) prevails over good states, namely those for which $\tilde{v}_t > \mathbb{E}\{\tilde{v}_t\}$, and that a positive value (2.434) prevails in bad states, namely those for which $\tilde{v}_t \leq \mathbb{E}\{\tilde{v}_t\}$. This gives clear indication on the state-contingent nature of the ambiguity attitudes, being averse to entropy deviations in bad states and opportunistic toward them in good states. According to Lemma 3.2 above we know that $\theta_t < 0$, which prevails in good states, implies that agents act optimistically. Similarly a $\theta_t > 0$, which prevails in bad states, speaks in favour of pessimism.

To further test our result above we run unconditional estimation over two different historical periods. We choose the first to be Great Moderation sample (1985:Q1-2007:Q2), which captures the boom phase preceding the 2007-2008 financial crisis. The sub-sample representing the recessionary states is the period following the crisis, namely the (2007:Q3-2016:Q4). Estimations, reported in the last two rows, confirm the same state-contingent nature uncovered in the conditional estimates. Finally note that for each sample reported the J-test fails to reject model in equation (24) at conventional significance levels.

Table 2: Estimated Moments of the Pricing Kernel

Moments (1980-2016)	$\Lambda_{t,t+1}$	$\Lambda^1_{t,t+1}$	$\Lambda^2_{t,t+1}$
Mean SDF	0.860	0.977	0.8803
Standard deviation SDF	40.1	0.53	40.9
$Corr(SDF, \Delta c_t)$	-0.138	-0.999	-0.12
$Corr(SDF, R^s_{t+1})$	-0.121	-0.332	-0.115

Next, given the estimated preference parameters we investigate the cyclical properties of the pricing kernel, namely the estimated SDF, and through them, those of the risk premia. To this purpose we use the decomposition of the SDF in $\Lambda^1_{t,t+1}$ and $\Lambda^2_{t,t+1}$ in order to isolate the contribution arising from the ambiguity attitudes. The empirical moments of the SDF are listed in table 2. They interestingly show that the high volatility in the SDF is totally driven by the ambiguity attitudes component, which, for the same reason, contribute less to the SDF clear countercyclical properties.

Given the above estimation results the process for θ_t reads as follows:

$$\theta_t = \begin{cases} \theta^- & \text{if} & V_t > \mathbb{E}_{t-1}\{V_t\} \\ \theta^+ & \text{if} & V_t \le \mathbb{E}_{t-1}\{V_t\} \end{cases} \tag{25}$$

We will use this process structure since now on.

5   Analytical Results

In this section we derive analytical expressions for asset price, premia and Sharpe ratio and show their dependence on the optimal LR and the shadow price of debt, μ_t. The analytical derivations will allow us to gain first economic intuition on the combined role of occasionally binding constraints and ambiguity attitudes for asset prices and leverage.

5.1   The Impact of Ambiguity on Asset Prices

Proposition 5.1 (Asset Price Recursion). *The recursive formula for the asset price over the infinite horizon in our model reads as follows:*

$$q_t = \lim_{T \to \infty} \mathbb{E}_t \left\{ \sum_{i=1}^{T} d_{t+i} \prod_{j=1}^{i} K_{t+j-1,t+j} \right\} \tag{26}$$

where $K_{t,t+1} = \frac{\Lambda_{t,t+1}}{1-\phi\mu'_t}$ *with* $\Lambda_{t,t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}$ *and* $\mu'_t = \frac{\mu_t}{u_c(c_t)}$.

Proof is described in Appendix B.1. The asset price clearly depends upon the optimal LR, m_{t+1}, and the shadow price of debt, μ_t. Consider first good states. In this case endogenous beliefs are right skewed toward the upper tails according to Lemma 1, hence both $\Lambda_{t,t+1}$ and $K_{t,t+1}$ are higher than when $m_{t+1} = 1$ for all positive states. In good states the asset price grows, due to increase asset demand, but it does so more under optimist beliefs. Similarly in bad states endogenous beliefs are left-skewed toward the lower tails, hence both $\Lambda_{t,t+1}$ and $K_{t,t+1}$ are higher than in the case with no ambiguity for all negative states. Asset price falls, but they do more so with pessimism. This is the sense in which ambiguity attitudes contribute to the heightened dynamic of the asset price boom and bust cycles. The asset price also depends upon the shadow price of debt, which proxies the margin or the down-payment requested to borrowers. When the constraint is binding margins are positive and increasing, in line with

empirical observations (see Geanakoplos (2010)). The higher margins paid by borrowers or the higher collateral value of the asset is reflected in higher asset prices. This also contributes to heightened asset price dynamics.

Proposition 5.2 (Equity Premium). *The return for the risky asset reads as follows:*

$$\mathbb{E}_t\{R_{t+1}^s\} = \frac{R(1 - cov(\Lambda_{t,t+1}, R_{t+1}^s) - \phi\mu_t^{'})}{1 - \mu_t^{'}} \tag{27}$$

while the premium of its return over debt return reads as follows:

$$\Psi_t = \frac{1 - cov(\Lambda_{t,t+1}, R_{t+1}^s) - \phi\mu_t^{'}}{1 - \mu_t^{'}}. \tag{28}$$

where $\Lambda_{t,t+1} = \beta\frac{u_c(c_{t+1})}{u_c(c_t)}m_{t+1}$ *and* $\mu_t^{'} = \frac{\mu_t}{u_c(c_t)}$.

See Appendix B.2 for the proof. The above proposition also shows unequivocally the dependence of the premia over the beliefs as captured by m_{t+1} and the shadow price of debt. While the exact dynamic of the equity premium depends on the solution of the full-model and upon its general equilibrium effects, we can draw some general conclusions on the dependence of the equity premium upon the beliefs and the shadow price of debt.

First, a negative covariance between the SDF and the risky asset returns implies that borrowers are less hedged. This results in a higher return required to hold the risky asset. The opposite is true for positive covariances. While we cannot say with certainty the sign of the $cov(\Lambda_{t,t+1}, R_{t+1}^s)$[21], we can conjecture that optimism and pessimism increase the covariance between consumption and asset returns. One way to see this is by looking at the upper bound for this covariance. By the Cauchy-Schwarz inequality that $cov(\Lambda_{t,t+1}, R_{t+1}^s) \leq \sqrt{Var(\Lambda_{t,t+1})Var(R_{t+1}^s)}$. Therefore anything that either increases the variance of $\Lambda_{t,t+1}$ or R_{t+1}^s will increase their covariance, whether in the positive or the negative domain. Endogenous beliefs formation by inducing fluctuations in m_{t+1} tend to increase the variance of the stochastic discount factor which is given by $Var(\Lambda_{t,t+1}) = Var\left(\beta\frac{u_c(c_{t+1})}{u_c(c_t)}m_{t+1}\right)$. Hence the variance of m_{t+1} adds up to the variance of the stochastic discount factor.

Second, the premium also depends upon the shadow price of debt. Taking as given again the covariance between the SDF and the risky return, one can compute the following derivative: $\frac{\partial\Psi_t}{\partial\mu_t^{'}} = \frac{(1-\phi)-cov(\Lambda_{t,t+1}, R_{t+1}^s)}{(1-\mu_t^{'})^2}$. If the $cov(\Lambda_{t,t+1}, R_{t+1}^s)$ is negative the derivative is certainly negative[22]. In other words when there are low hedging opportunities a tightening of the constraint implies that borrowers require higher premia to hold the risky asset. The asset already conveys poor insurance opportunities, a tightening of the constraint by reducing the asset collateral value, reduces its demand. Hence borrowers are willing to hold only at higher premia. Endogenous beliefs also play an indirect role in this dependence. Indeed as explained above fluctuations

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[21]This indeed depends on whether $\mathbb{E}_t(\Lambda_{t,t+1}, R_{t+1}^s) > \mathbb{E}_t(\Lambda_{t,t+1})\mathbb{E}_t(R_{t+1}^s)$ or $\mathbb{E}_t(\Lambda_{t,t+1}, R_{t+1}^s) < \mathbb{E}_t(\Lambda_{t,t+1})\mathbb{E}_t(R_{t+1}^s)$.

[22]If the $cov(\Lambda_{t,t+1}, R_{t+1}^s) > 0$, then whether $\frac{\partial\Psi_t}{\partial\mu_t^{'}}$ is positive or negative depends upon whether the $cov(\Lambda_{t,t+1}, R_{t+1}^s) > (1 - \phi)$ or not.

in beliefs generally raise the absolute value of the covariance. Hence, consider again the case of a negative covariance. In this case fluctuations in beliefs impair even more the hedging abilities of the risky assets and this in turn increases the premium that borrowers ask in face of a tightening of the borrowing limit.

Proposition 5.3 (Sharpe Ratio). *The Sharpe ratio in our model reads as follows:*

$$SR = \frac{\mathbb{E}_t\{z_{t+1}\}}{\sigma_z} = \left[\frac{\sigma_{\Lambda_t^*}^2}{\bar{\Lambda}^{*2}} - 2\mu_t' \frac{(\phi-1)\mathbb{E}_t\{z_{t+1}\}}{\sigma_z^2} - \frac{\mu_t'^2}{\bar{\Lambda}^{*2}} \frac{(\phi-1)^2}{\sigma_z^2} \right]^{\frac{1}{2}} \tag{29}$$

where $z_{t+1} = R_{t+1}^s - R$ is the asset excess return $\bar{\Lambda}$ is the long run value for the SDF, $\sigma_{\Lambda_t^*}^2$ is the volatility of the SDF and σ_z^2 is the volatility of the excess return.

Proof is given in Appendix B.3. The presence of endogenous beliefs raises the Sharpe ratio and brings it close to the empirical values as we show in Table 4. Matching the empirical values of the Sharpe ratios is typically hard for models with asset pricing and/or financial frictions. The reason being that typically an increase in the excess returns of the risky assets is accompanied by an increase in its volatility. Analytically it is easy to see why the Sharpe ratio raises in our model. First fluctuations in m_{t+1} raise fluctuations in the stochastic discount factor, Λ_t^*, hence in its variance. This in turn raises the Sharpe ratio. Second, fluctuations in θ_t enhance fluctuation in beliefs, m_{t+1}. Third, the kinked nature of the value function steepens fluctuations in m_{t+1} and the SDF also since marginal utilities tend to infinity around the kink. In turn any increase in the variance of m_{t+1} raises the variance of Λ_t^* and the Sharpe ratio. Intuitively in presence of *uncertainty* or ambiguity agents require a premium which goes beyond the one needed to cover risk[23] as measured by the volatility of the excess return. If agents knew the objective probability distribution, they would need to be compensated only for bearing tail risk. As the tail itself is uncertain, borrowers require a higher premia.

In past literature it was noted that the model implied Sharpe ratio can match the empirical counterpart by assuming implausibly large values for the risk-aversion parameter (see Cochrane (2005), chapter 13). In the numerical simulations below we show that this is not the case for our model.

At last, note also that the Sharpe ratio depends negatively upon the shadow value of debt. When the constraint binds borrowers start to de-leverage and to reduce the demand of risky asset. As a result this reduces the expected excess returns relatively to the return on debt. This is compatible with the pro-cyclical nature of the returns on risky assets observed in the data.

6 Quantitative Results

In this section we solve the model numerically employing a global solution method, namely policy function iterations with occasionally binding constraints. We provide details on the solution method in Appendix C. We group our results in three. First, we search for the optimal

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[23]Here we refer to the distinction between uncertainty and *risk* introduce by Knight (1921).

model calibration. To do so we choose some target moments in the data and we search for the set of parameters that minimizes the distance between the targets and the model-implied moments. This gives further empirical validation of our model. Second, under the optimal calibration we verify if the model can match several volatilities and correlations for asset prices, returns, equity premia and leverage. We show that in fact the model does it well. At last, under this optimal calibration we examine policy functions and we conduct a crisis event exercise. Our main result is that with ambiguity the model produces steeper asset prices and leverage increases in booms, which are then followed by sharper de-leverage and crises in recessions.

6.1 Calibration Strategy

This section describes the calibration strategy. We divide the set of structural parameters in three groups. The first group includes parameters which are calibrated using external information. Those are the risk free rate, the loan-to-value ratio, the fraction of financial wealth over total wealth. The second group includes parameters calibrated using a matching moments routine. Those are θ, the absolute risk aversion coefficient, the discount factor and the volatility of the income process. The third group includes parameters which are calibrated with the estimation of the income process, more specifically the autocorrelation of the income process. Table 3 summarizes the results of the calibration procedure.

In order to calibrate the second group of parameters, we choose to match six empirical moments (the matching is shown in Table 4, where also other moments are displayed), namely the volatility of debt σ^b, the autocorrelation of debt ρ^b, the correlation between debt and consumption $Corr(\Delta b^t, \Delta c^t)$, the expected return on risky asset $\mathbb{E}_t(R_t^s)$, the volatility of return on risky asset σ^{R^s}, the correlation between return on risky asset and consumption growth $Corr(R_t^s, \Delta c_t)$. To compute the empirical equivalent we focus on the data sample 1980:Q1-2016:Q4, which captures a period of both of large debt growth and subsequent de-leverage. More details on the data sources are in Appendix D. We do not include the equity premium among our targets because the risk free rate is exogenous in the model, but we show later on that our model can match it well. Note that while the income shock correlation is directly estimated in the data, its volatility is instead calibrated. It is indeed well known from past literature that estimated values exhibit large measurement errors (see Heaton and Lucas (1996) and Deaton (1991)).

Table 3: Values for the calibrated parameters

Parameter	Meaning	Strategy	Value
R	*Risk-free rate*	3month T-bill rate	1.0114
ϕ	*Loan-to-value ratio*	Crises Probability	0.15
α	*Share of dividend*	Fraction of financial wealth	0.10
$\theta^{+}(V_t \leq \mathbb{E}_{t-1}\{V_t\})$	*Pessimism*	Matching Moments	+1.35
$\theta^{-}(V_t > \mathbb{E}_{t-1}\{V_t\})$	*Optimism*	Matching Moments	-1.35
γ	*Risk aversion*	Matching Moments	2.075
β	*Discount factor*	Matching Moments	0.930
σ_y	*Income Volatility*	Matching Moments	0.0415
ρ_y	*Income Persistence*	Estimation	0.634

The matching moment routine starts from the following grids: $\sigma^y \in [0.02, 0.07]$ for the states of the income shock, $\beta = [0.92, 0.98]$, $\gamma = [1, 2.2]$, and finally $\theta_t \in \{[-5, 5], 100\}$[24]. In the grid for θ_t we introduce the value 100, in order to check if the model with no ambiguity produces theoretical moments which perform better than our model with waves of optimism and pessimism. Moreover, the grid is defined between 5 and -5 because out of these bounds the difference between the model with and without ambiguity becomes negligible.

It is interesting to note that the estimation of the full model through moments matching equally delivers the same type of state-contingent process for the parameter θ_t as the one we uncovered with our GMM estimation above. The estimated values are naturally different between the two estimation methods, since in the GMM case the regression is based on one equation summarizing only borrowers' first order conditions, while in he second case the estimation involves the full set of model equations. But the fact that the two estimations deliver the same type of state-contingent process is important.

6.2 Empirical Moments Matching

In this section we evaluate the model's ability to match the empirical moments under the optimal calibration determined above. We also compare the theoretical moments of our model with ambiguity attitudes (labelled AA since now on) with those of the equivalent model but with rational expectation (labelled RE since now on). The following Table 4 summarizes the main results:

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[24]For each parameter we check that the optimal values do not hit the bounds of the grid.

Table 4: Empirical and model-based moments

Moments	Mnemonics	Empirical	Model AA[1]	Model RE
Matched Moments				
Volatility debt	σ^b	12.52	12.37	7.24
Persistence debt	ρ^b	0.846	0.539	0.331
Cyclicality debt	$Corr(\Delta b_t, \Delta c_t)$	0.668	0.378	0.821
Exp risky return	$E_t(R_t^s)$	9.38	8.19	7.38
Volatility risky returns	$\sigma^{R_t^s}$	16.21	17.46	12.40
Cyclicality risky returns	$Corr(\Delta R_t^s, \Delta c_t)$	0.474	0.989	0.989
Other Relevant Moments				
Equity premium	$E_t(R_t^s - R)$	8.25	7.05	6.24
Sharpe ratio	$\frac{E_t(R_t^s - R)}{\sigma^{R_t^s - R}}$	0.522	0.404	0.503
Prob(crisis)	-	4.00[3]	3.16	4.51

[1] *Column 2 and 3 compare theoretical moments under ambiguity versus rational expectation;*
[2] *We do not calculate the empirical frequency of the financial crises but we follow Bianchi and Mendoza (2015), who derive an average of 4 crises every 100 years in the developed countries.*

The upper panel of Table 4 shows the matched moments (according to the criteria set in the previous section), while in the lower panel other relevant moments are shown. The overall message is that our model fits well the empirical moments. First, it is better capable of matching empirical debt and risky asset return volatilities, relatively to the RE model. This is so despite both models exhibit amplification induced by the occasionally binding collateral constraint. This shows that endogenous beliefs are also needed to explain asset and debt markets dynamics. The equity premium as well as its cyclical properties are also well captured and again the presence of ambiguity attitudes seem to improve even above the benchmark model featuring solely the collateral constraint. As explained in Cochrane (2005) the ability to match contemporaneously the long run equity premia and asset returns and their cyclical properties is related to the agents' attitude toward events on the tails. In our model borrowers are endogenously optimistic, hence risk-takers, on the upper tail, while they are pessimistic, hence risk-sensitive, on the lower tail. This additional effect, stemming from the endogenous waves of optimism, improves the ability of the model to match the equity premium and its cyclical properties. In terms of matching the Sharpe ratio, both models seem to perform similarly and with acceptable performance. This in part shows that the kink induced by the occasionally binding collateral constraint is enough to get close to the empirical Sharpe ratio. However, one important aspect shall be noted. In the RE model the Sharpe ratio is matched since the volatility of excess returns (hence the denominator of the Sharpe ratio) is low, while in the AA model the Sharpe ratio is matched despite the high volatility of excess risky returns. At last, both model match the pro-cyclicality of leverage which is well documented in the data. Leverage indeed increases in booms due to a combination of exuberance and lax debt constraints and declines in recessions due to a combination of pessimism and increasing margins, namely borrowers' down-payments. Here

neither our model nor the RE model seem to match the empirical value with precision, as the first underestimates, while the second overestimates.
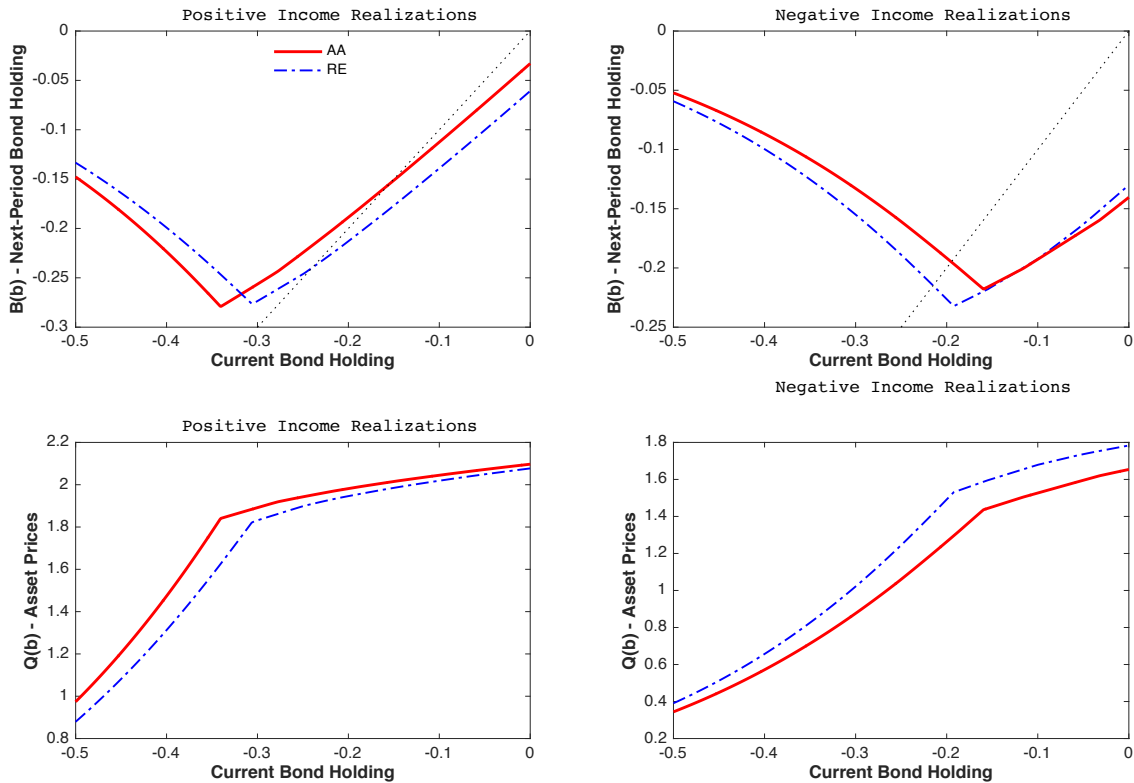
At last, note that the model reasonably matches the empirical probability of the crises, for which we rely on the value presented in Bianchi and Mendoza (2015).

6.3 Policy Function and Crisis Event

We have argued that our leverage cycle model with state-contingent beliefs' distortions has a sound empirical ground. The estimated SDF implied by our model shows that the role of ambiguity attitudes is significant and sizeable. Under the estimated ambiguity parameter and the empirically optimal calibration, we also showed that our model can account well for several asset price and leverage moments. This second exercise serves a cross-check of the model empirical validity.

Given the above, we proceed describing the dynamic properties of our model in comparison to the RE benchmark and by focusing in particular on the leverage and asset price cycles and on the unfolding of a crisis. We do so in two steps. First, we plot policy functions of debt and asset prices. Next, we simulate a crisis event.

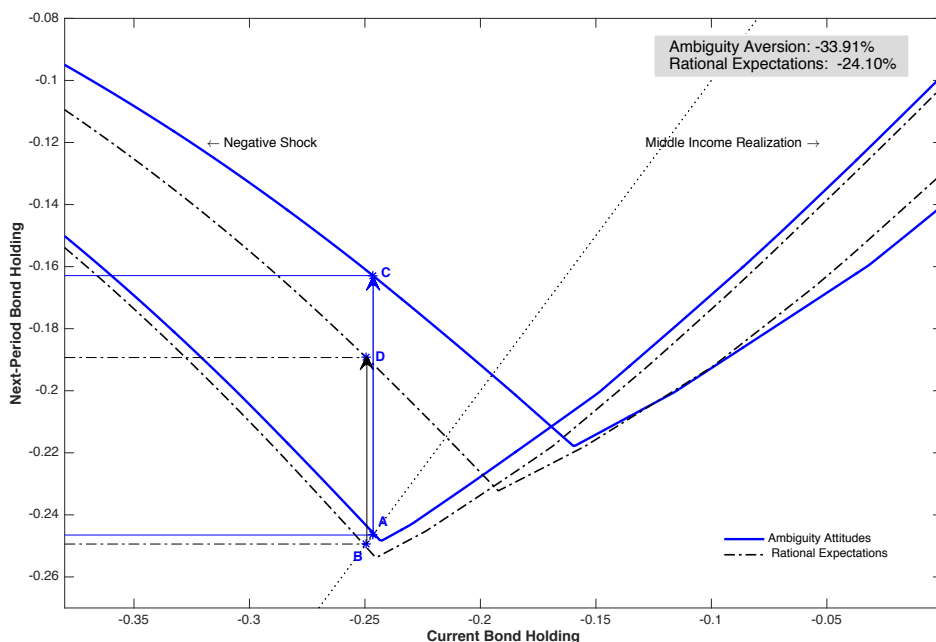Figure 1: Policy functions for debt and asset prices

6.3.1 Borrowing and Asset Pricing decisions

Figure 1 below shows the decision rules for debt and asset prices with respect to past debt holdings across the model with ambiguity attitudes, labelled AA (red line) and the model with

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rational expectations, labelled RE (blue line). Note that the full set of policy functions can be found in Appendix E. We interpret the results distinguishing between positive (+5% from income trend; left panels) and negative realizations of the shock (−5% from trend; right panels) in order to appreciate the non-linearity arising by the changing ambiguity attitudes over the different states of the economy. Moreover, in each panel the kink separates the constrained from the unconstrained region and it represents the point at which the collateral constraint is marginally binding in each economy. Finally, the intersection between the 45 degree line and the policy function defines the stationary levels of debt. Several considerations emerge. First, both economies are able to produce the V-shaped bond holdings decision rules, which are a typical feature of models with high deleveraging and financial crises (see e.g. Bianchi (2011) and Bianchi and Mendoza (2015)). To the right of the kink the policy functions are upward-sloping, corresponding to the unconstrained values of debt, while to the left they are downward-sloping identifying the constrained region where next-period bond holdings decrease in current bond holding. The kinked policy functions for asset prices follow accordingly: they increase with wealth and more steeply in the constrained region.

Figure 2: Debt Amplification Dynamics

Second, the policy function for the AA model moves away from the one under RE both in the scale of the dynamics in each region and in the position of the kinks. In particular, given a negative state of the economy, higher previous-period debt induces a binding constraint earlier, increasing the probability of lying in the financial amplification region. The opposite holds for booms, where optimism boosts the collateral values, which in turn relaxes the constraint and facilitates the build-up of leverage. Thus, given the shifted location of the binding and slack regions, debt choices under AA, when constrained, associate a sharper or a more damped contraction in debt whether the economy is in booms or in busts. This nonlinearity reflects

optimistic and pessimistic attitudes toward future realizations and generates amplification dynamics in the leverage cycles. We will visualize the size of this result below. At last, focusing on the asset price panels the comparison between the two models turns to be quite interesting. Asset prices in the AA model lie always above the RE benchmark in booms and always below in busts, which is coherent with the ability of the AA model to associate to a given initial debt position more debt and less debt, respectively for the two income states. Next we compute how large would be the extent of a de-leverage when the steady state of the economy is perturbed by a one-period 5% fall in income. This exercise offers a clear visualization of the enhanced financial amplification dynamics produced by AA, keeping the parallel with the RE model.
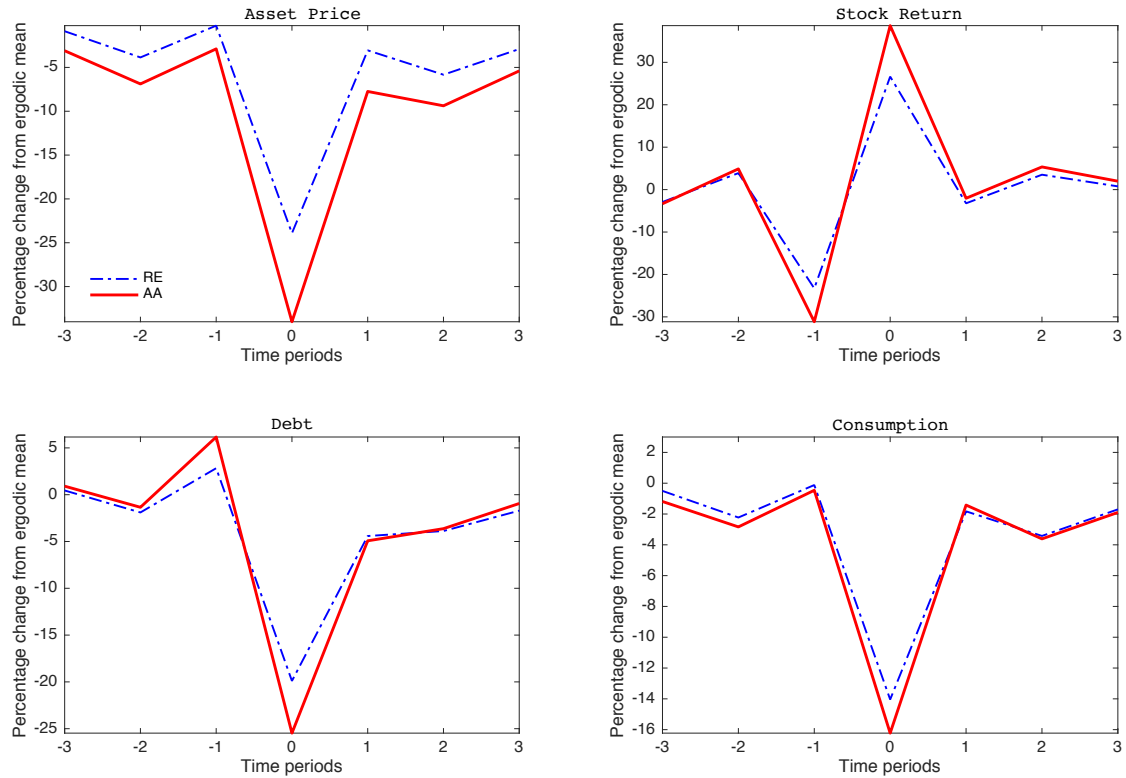
Figure 2 reproduces the following experiment. Assume that the two economies lie in equilibrium in A and B, respectively. Then, at the time of the shock the new negative realization of income forces a sharp upward adjustment of the bond decision rules and the temporary equilibria jump to C and D. The arrows define a drop in bond holdings which results to be much more pronounced for the model under the AA model. Interestingly, the AA model generates a drop of -33.9%, which exceeds the RE equivalent by about 10 points. This speaks about the model's quantitative relevance in producing amplified leverage cycles.

6.3.2 Financial Crises

The crisis event displayed in Figure 3 proves the model's ability to generate financial crises and studies relevant macro dynamics around it. More in detail, the event analysis is realized using model-simulated data for the two economies, AA and RE, and defining as crises the events in which the collateral constraint binds and the current account is at least two standard deviations above the trend. Then, we construct seven-periods event windows centred on the crisis to analyse pre- and post-crises patterns.

From the comparison between the two economies lies in the ability of the model with AA to account for stronger build-up of leverage prior to the crisis (around +3%) and sharper de-leveraging at the crisis (around -7%). Again the role of the state contingent distortion is important in understanding this dynamic. In booms optimism boosts collateral values, relaxing the constraint and facilitates the build-up of leverage. In recessions the opposite is true. Pessimism induces assets' fire sales, this generates sharper declines of the collateral values forcing borrowers to de-leverage earlier and more severely. Accordingly, looking at asset prices, consumption and equity returns helps understanding the results around debt decisions. Indeed, all of them display more severe dynamics under ambiguity aversion. The asset price collapses, for instance, playing an important role in explaining the more pronounced decline in debt under the AA model, reflecting a strong Fisherian deflation mechanism. Moreover, consumption falls 2 percentage points more and the risky return results to drive the enhanced pre- and post-crisis debt patterns, falling more sharply in booms and increasing when the crisis occurs.

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Figure 3: Crises Event Study

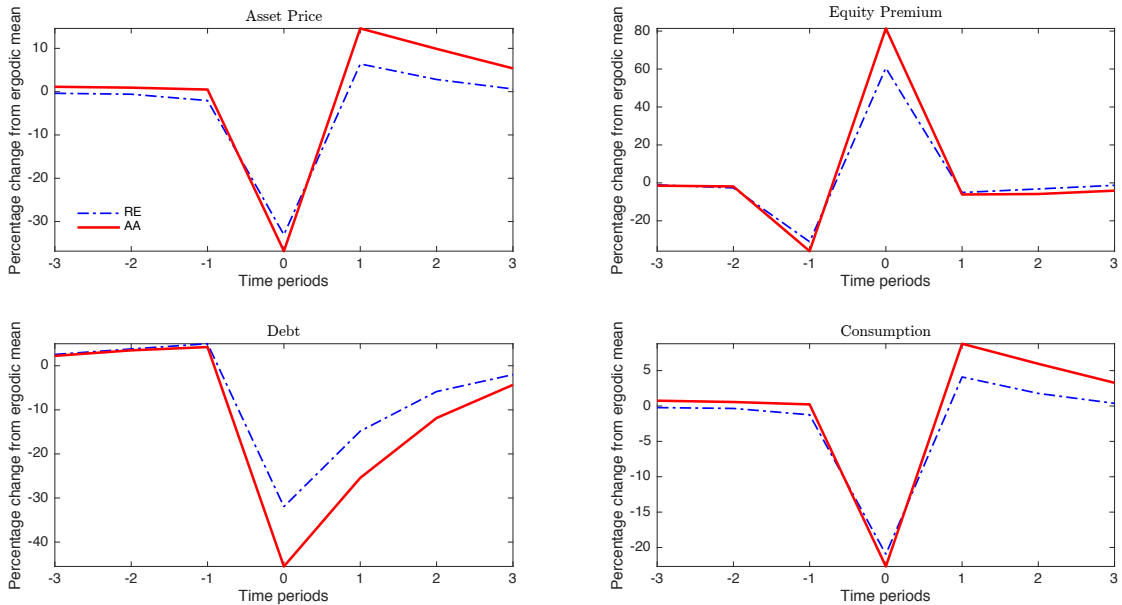## 6.4 Intermediation Sector and Intermediation Shocks

Lack of transparency and ambiguity play an important role in crises developments as we showed so far, but by no means instability stemming from the intermediation sector, hence originating in the credit supply, has a major role too. This is particularly true within the context of the 2007-2008 financial crisis. While including all possible sources of intermediation disincentives is beyond the scope of this paper, we nevertheless wish to assess the role of the intermediation channel. This is important as one should test whether the beliefs-related channels described so far persist even when the supply side of credit is inserted in the model. In fact, we find that not only the role of ambiguity attitudes is preserved, but in most cases is amplified and the interaction with the intermediation channels is compelling.

We introduce intermediation by assigning the role of debt monitoring to a bank. This is actually realistic since atomistic lenders do not monitor or screen debtors individually, but largely assign this function to an intermediary. In this context the collateral constraint results from the bank design of a debt contract that is incentive compatible, meaning that it reduces the incentives of the borrower to divert resources and default. We formalize this type of contracts and show how the collateral constraint emerges from such incentive compatibility constraint in Appendix G Within this context an intermediation shock, which suddenly tightens the supply of credit, affects the parameter governing the loan-to-value ratio, ϕ, which itself governs the strength of the incentive problems. Intuitively the shock can be interpreted in two ways, both affecting the contractual agreement in a similar vein. It could capture financial innovation in

the form of derivatives and/or asset back securities issuance, which being pervasive prior to the crisis, allowed banks to off-load credit risk and reduced the tightness of the debt contract. A sudden freeze of the asset backed market liquidity due for instance to the sub-prime shock would have then induced a sudden fall in ϕ. A second interpretation, linked to the first, is that higher availability of liquidity[25] prior to the crisis had lessened banks' monitoring incentives, something which resulted in higher loan-to-value ratios, ϕ. After the crisis occurs, the squeeze in liquidity, hence banks' funding, could suddenly tightens the loan-to-value ratio. Both interpretations, which are realistic particularly in the context of the recent financial crisis, have the effect of producing a sudden tightening of credit supply. Within this context we subject our model to an intermediation shock to ϕ and assess its role as well as its interaction with ambiguity attitudes. We do so by analysing again policy functions, crisis events and second moments of the model.

Before proceeding to the assessment of the quantitative results, a few words are needed regarding the calibration of the shock. We define a high and a low level of the loan-to-value ratio, respectively $\phi_l = 0.22$ and $\phi_h = 0.28$, calibrated in order to match the empirical volatility of debt. The shock then follows a two-state regime-switching Markov process, with a transition matrix calibrated to replicate the empirical probability and duration of the crises events, as in Bianchi and Mendoza Bianchi and Mendoza (2015). More in detail, the probability to remain in a high state, π_{hh} is set equal to 0.955 in order to match a frequency of crises close to 4%, while the transition probability from a low to high state π_{lh} is equal to one, implying a one year duration of the crises. The remaining transition probabilities are set as complements of the previous ones, i.e $\pi_{hl} = 1 - \pi_{hh}$ and $\pi_{ll} = 1 - \pi_{lh}$.

Figure 4: Crises Event Study with income and intermediation shock

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[25]This again could be due either to the possibility of raising additional bank liabilities through asset backed securities or through the ample availability of liquidity in interbank and repos markets prior to the 2007-2008 crisis.
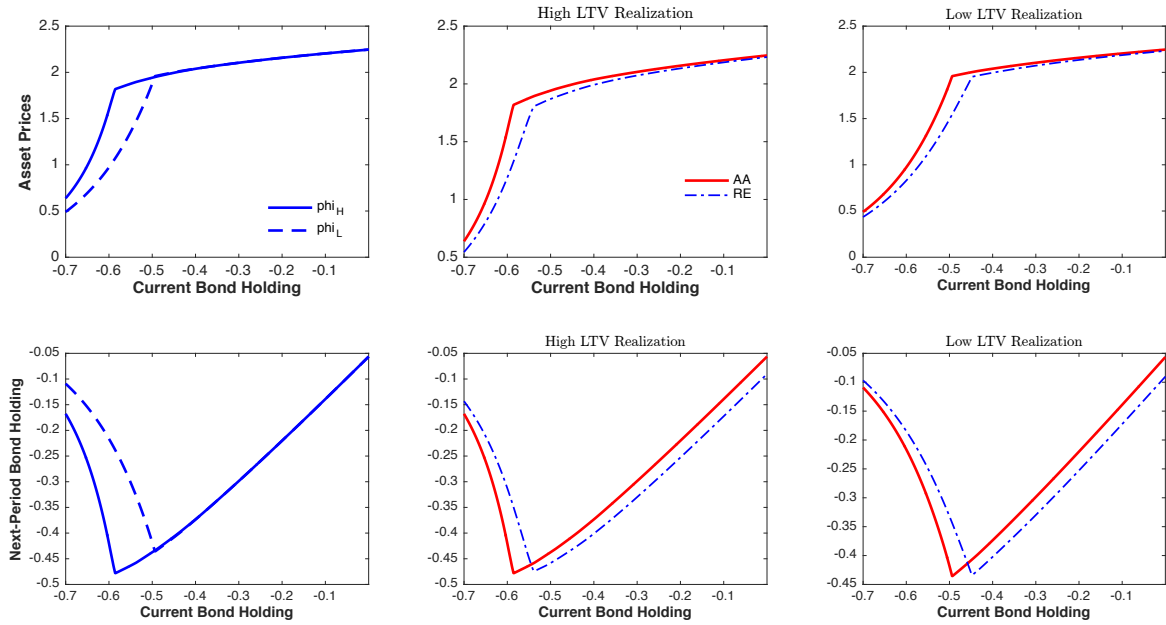
We start in this case from a crisis event, since this makes immediately visible the role of the credit supply for the crisis development on top of the role of ambiguity attitudes. Figure4 compares the crisis event in the model with ambiguity attitudes and with rational expectations. The crisis event is defined as before, but now it is triggered by a combination of income and intermediation shocks. Specifically, we simulate the model in response to both shocks, we then observe that the crisis originates exactly when both shocks turn negative. The Figure shows two interesting facts. First, the role of ambiguity attitudes remains. It is still true that beliefs formation by affecting the value of collateral through endogenous skewed beliefs induce sharper crises than under the case with no ambiguity. Second and interestingly, this time the drop in the crisis is even larger. This is reasonable since now both credit demand side and supply side components are operative. Intuitively the steepness of the crises now depends on two channels. As before the positive skewed beliefs, valid prior to the crisis, induced higher demand for leverage and the negative skewed beliefs, materializing after the crisis, induce de-leveraging. On top of this the progressive reduction of ϕ facilitates debt supply prior to the crisis and produces a credit crunch after the crisis.

To examine more in details the intermediation channel we examine the policy functions for debt and asset prices. Figure 5a below shows the policy functions conditional to positive realizations of the income shock for asset prices and debt by comparing various scenarios. In the first column we compare the model with ambiguity attitudes for two values of ϕ. This case allows us to isolate only the contribution of credit supply. As before the kink represents the turn in which the constraint shifts from binding to non-binding. The comparison shows that a low ϕ, namely tight credit due to high monitoring standards or low availability of liquidity, has two effects. On the one side, it enlarges the constrained region. On the other side, it reduces leverage, and this effect can be beneficial in the medium to long run. The second and the third columns compare the models with and without ambiguity attitudes, respectively for low levels of ϕ (second column) and high levels of ϕ (third column). Two interesting observations emerge. First, as before under the model with ambiguity attitudes asset prices are higher and debt displays the previously underlined nonlinear dynamics over constrained and unconstrained regions. This as before is due to the nature of the positive skewed beliefs that emerge under positive income shocks. Second, the comparison between a high and a low level of ϕ shows that the qualitative pattern of the policy functions remains unaltered, albeit the constrained region is expanded under the low loan to value ratio. In other words, the forces operating through the ambiguity channel remain active even when introducing supply side elements. The dominant effect of the latter is more evident in terms of changes in the size of the constrained region. To fully complete the assessment of the policy functions Figure 5b shows the results for the policy functions conditional on negative income realizations. The message is largely symmetric to the one described above.
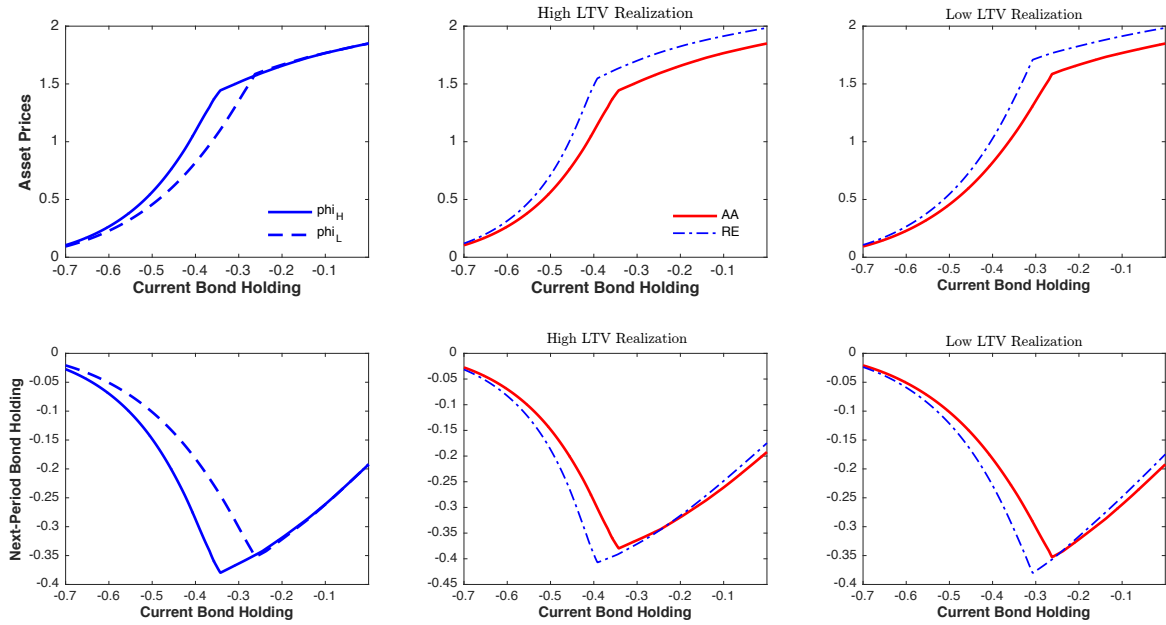
At last, we ask whether the introduction of the intermediation shock can improve upon the moment matching and if so along which dimension. Table5 below shows again the comparison of a selected numbers of second moments between the data, the model with and without ambiguity attitudes. This time the comparison is done by simulating the model also in response to the

Figure 5: Policy Functions for the model with intermediation

(a) Positive intermediary shock realization

(b) Negative intermediary shock realization

intermediation shock. The addition of the intermediation shock preserves most of the previous moments and improves in terms of data matching along other dimensions. The Table highlights primarily moments that change with the introduction of the intermediation shock. The most noteworthy result is that the introduction of credit supply fluctuations increases debt pro-cyclicality, which as discussed before, is an important stylized fact. The reason is intuitive. The double occurrence of the negative income and credit supply shock tightens leverage much more sharply. Equally the double-coincidence of the positive income and credit supply realizations heightens the build-up of leverage. Those movements on the tails help to increase average pro-cyclicality. The volatility of debt is also somewhat higher, mostly so in the model with ambiguity attitudes, and is closer to the data value. This again might be due to the contribution of the tails. On the other side, it shall be mentioned that the introduction of the intermediation shock worsens the volatility of risky returns, which now goes above the one detected in the data. This effect is possibly due to the fact that our model does not account for loss absorption capacity of the intermediation sector in terms of equity capital and/or liquidity buffers. Those elements would indeed limit the extent of fire sales in risky assets when credit supply tightens, hence they would reduce fluctuations in asset prices.

Table 5: Selected empirical and model-based moments

Moments	Mnemonics	Empirical	**Model AA**	**Model RE**
Matched Moments				
Volatility debt	σ^b	12.52	11.55	9.78
Persistence debt	ρ^b	0.846	0.432	0.385
Cyclicality debt	$Corr(\Delta b_t, \Delta c_t)$	0.668	0.792	0.795
Exp risky return	$E_t(R_t^s)$	9.38	8.67	7.88
Volatility risky returns	$\sigma^{R_t^s}$	16.21	23.45	19.40
Cyclicality risky returns	$Corr(\Delta R_t^s, \Delta c_t)$	0.474	0.983	0.992
Equity premium	$E_t(R_t^s - R)$	8.255	7.013	7.050
Prob(crisis)	-	4.0	4.06	5.53

To sum up the main contribution of the intermediation channel in our model is that of modifying the size of the constrained versus the unconstrained region, that of contributing to explain the severity of a financial crisis and that of contributing to explain debt pro-cyclicality.

7 Conclusions

Financial crisis are most often triggered by endogenous instability in debt markets. The latter are typically characterized by collateral constraints and opacity in asset values. Under lack of transparency the beliefs formation process acquires an important role since eventually it affects the value of collateral and with it the debt capacity. The narrative of most crises depict sharp increases in debt and asset prices prior to them and sharp reversal afterwards.

We therefore introduce in a model in which borrowers fund risky assets through debt and

are subject to occasionally binding collateral constraints, beliefs formation, driven by ambiguity attitudes that endogenously induce optimism in booms and pessimism in recessions. In booms optimistic borrowers demand more risky assets, which results in higher asset price growth (compared to the case with only collateral constraints), and lever up more. In recessions pessimistic borrowers de-leverage sharply and off load risky assets. This beliefs formation process coupled with the occasionally binding nature of the collateral constraint is a crucial element in explaining the combined amplified dynamic of asset prices and leverage as well as the whole span of their long run and short run statistics. Importantly we assess the empirical validation of our model both through GMM estimation of the Euler equation and through data-model moment matching.

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A GMM Estimation of the Ambiguity Parameter

In this section we detail the derivations needed to achieve the moment condition that is the object of our estimation. Further below we also provide a description of the dataset used in the estimation.

A.1 General Approach

We use a GMM estimation procedure based on the moment condition obtained from the combined Euler equation for debt and risky assets and is a variant of the techniques developed for asset pricing models with recursive preferences, pioneered by Epstein and Zin (1989) and Kreps and Porteus (1978). Hence, the starting point is to reformulate our value function, capturing multiplier preferences, in terms of an ambiguity term. The latter is achieved by mapping the multiplier preferences to a special case of the recursive preferences. This can be done by assuming a logarithmic continuation value, a logarithmic utility function and an ambiguity adjustment factor, Q which accounts for waves of optimism and pessimism. Indeed we depart from the well-known equivalence between multiplier and recursive preferences by embedding state-contingent ambiguity attitudes. We start by reporting the value function derived after substituting the solution of the inner problem, presented in Section 3.4.1:

$$V_t = u(c_t) - \beta \theta_t log \left[\mathbb{E}_t \left\{ \exp \left(-\frac{V_{t+1}}{\theta_t} \right) \right\} \right] \tag{30}$$

The above equation embeds a logarithmic ambiguity-adjusted component $Q_t(V_{t+1})$, which maps future continuation values into current realizations. Indeed we can re-write (30) as follows:

$$\begin{aligned} V_t =& u(c_t) + \beta h^{-1} \mathbb{E}_t \left\{ h(V_{t+1}) \right\} \\ & u(c_t) + \beta Q_t(V_{t+1}) \end{aligned} \tag{31}$$

where $h(V_{t+1}) = \exp \left(-\frac{V_{t+1}}{\theta_t} \right)$, as implied by the equivalence between specifications under recursive and multiplier preferences (see Hansen et al. (2007)). It then follows that the ambiguity adjustment component reads as follows:

$$Q_t(V_{t+1}) = h^{-1} \mathbb{E}_t \left\{ h(V_{t+1}) \right\} = -\theta_t \log \left[\mathbb{E}_t \left\{ \exp \left(-\frac{V_{t+1}}{\theta_t} \right) \right\} \right] \tag{32}$$

A.2 Pricing Kernel-SDF

The next step to obtain our moment condition is to derive an expression for the stochastic discount factor as function of $Q_t(V_{t+1})$. To this purpose, we shall derive the marginal utility of consumption and the derivative of the current value function with respect to the next period

one, which we define as MV_{t+1} and reads as follows:

$$MV_{t+1} = \frac{\partial V_t}{\partial Q_t(V_{t+1})} \frac{\partial Q_t(V_{t+1})}{\partial V_{t+1}} = \beta \frac{\exp(-\frac{V_{t+1}}{\theta_t})}{\mathbb{E}_t\left\{\exp(-\frac{V_{t+1}}{\theta_t})\right\}} \tag{33}$$

$$= \beta \exp\left(-\frac{1}{\theta_t}(V_{t+1} - Q_t(V_{t+1}))\right)$$

Given a logarithmic utility function $u(c_t) = log(c_t)$, the marginal utility of consumption is $MC_t = c_t^{-1}$. Using the above expressions we can derive the SDF as function of Q_t:

$$\Lambda_{t,t+1} = \frac{MV_{t+1}MC_{t+1}}{MC_t} = \beta \frac{c_{t+1}}{c_t}^{-1} \underbrace{\exp\left(-\frac{1}{\theta_t}(V_{t+1} - Q_t(V_{t+1}))\right)}_{m_{t+1}} \tag{34}$$

where $m_{t+1} = \exp\left(-\frac{1}{\theta_t}(V_{t+1} - Q_t(V_{t+1}))\right)$ is the optimal likelihood ratio. Equation (34) shows that the SDF has a two-factor structure. The first factor is the standard consumption growth, while the second is the ambiguity factor. The latter depends upon the distance between the future value function and its certainty equivalent, namely the future insurance premium. Under no uncertainty this premium vanishes[26].

A.3 Estimation of the Continuation Value Ratio

Since estimation requires strictly stationary variables, we shall re-scale the value function (31) by consumption (see Hansen, Heaton and Li (2008) (HHL henceforth). Subtracting the log of consumption, $\tilde{c}_t = log(c_t)$, on both sides:

$$\tilde{v}_t = \beta Q_t(\tilde{v}_{t+1} + \Delta \tilde{c}_{t+1}) \tag{35}$$

where we define $\tilde{v}_t = V_t - \tilde{c}_t$ as the *continuation value ratio*, scaled by the log of consumption. Next substituting (32) into (35) we obtain:

$$\tilde{v}_t = -\beta \theta_t \log(\mathbb{E}_t\left\{\exp\left[\sigma_t(\tilde{v}_{t+1} + \Delta \tilde{c}_{t+1})\right]\right\}) \tag{36}$$

where $\sigma_t = -1/\theta_t$, and it is negative when $\theta_t > 0$ and positive when $\theta_t < 0$. An expression for equation (36) can be derived analytically along the lines of HHL. Indeed, since $\tilde{v}_t$ is a function of states governing the dynamic behaviour of consumption growth, g_{t+1}^c, we can guess it as a function of a Markov process, ξ_t:

$$g_{t+1}^c = \tilde{c}_{t+1} - \tilde{c}_t = \mu_c + H\xi_t + \mathbf{A}\epsilon_{\mathbf{t+1}} \tag{37}$$

$$\xi_{t+1} = F\xi_t + \mathbf{B}\epsilon_{\mathbf{t+1}} \tag{38}$$

where ϵ_{t+1} is a $(2x1)$ i.i.d. vector with zero mean and covariance matrix I. A and B are $(2x1)$

---

[26]Indeed the continuation value would be perfectly predictable $\left(\exp\left(-\frac{V_{t+1}}{\theta}\right) = \mathbb{E}_t \exp\left(-\frac{V_{t+1}}{\theta}\right), m_{t+1}^* = 1\right)$ with zero adjustment $(Q_t(V_{t+1}) = V_{t+1})$.

vectors. The exogenous states, ϵ_{t+1}, income shocks in our case, have an impact on consumption directly and through the states, ξ_t. Its estimated value, $\hat{\xi}_t$, is obtained through Kalman filtering consumption data. Then, given (37), we guess the continuation value ratio as depending only upon the estimated states, $\hat{\xi}_t$:

$$\tilde{v}_t = \mu_v + U_v \hat{\xi}_t \tag{39}$$

where $U_v \hat{\xi}_t$ is the discounted sum of expected future growth rates of consumption. After some derivations we can write U_v and μ_v as follows:

$$U_v \equiv \beta(I - \beta F)^{-1} H \tag{40}$$

$$\mu_v \equiv \frac{\beta}{1 - \beta} \left(\mu_c + \frac{\sigma_t}{2} |A + U_v B|^2 \right)$$

where the term $A + U_v B$ captures the dependence between the the continuation value and the exogenous shocks.

A.4 SDF and the Euler Equation

Next, given the estimated $\tilde{v}_t$ from (39). Substituting (36) into (34) delivers:

$$\Lambda_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1} \left(\frac{\frac{\exp(V_{t+1})}{c_{t+1}} \frac{c_{t+1}}{c_t}}{\exp\left(Q_t\left(\tilde{v}_{t+1} + \Delta \tilde{c}_{t+1}\right)\right)} \right)^{\sigma} \tag{41}$$

Note that equation (41) is equivalent to the SDF obtained under Epstein and Zin (1989) preferences and given the assumption of unitary EIS. At last, upon using (35) into (41) and substituting the resulting SDF into the combined Euler for debt and risky asset (15) and (16), we obtain:

$$\mathbb{E}_t \left\{ \underbrace{\beta \left(\frac{c_{t+1}}{c_t} \right)^{-1} \left(\frac{\frac{\exp(V_{t+1})}{c_{t+1}} \frac{c_{t+1}}{c_t}}{\sqrt[\beta]{\left(\frac{\exp(V_t)}{c_t} \right)}} \right)^{\sigma_t}}_{\Lambda_{t,t+1}} \left(R_{t+1}^s - \phi R_{t+1} \right) + \phi - 1 \right\} = 0 \tag{42}$$

where $R_{t+1}^s = \frac{d_{t+1} + q_{t+1}}{q_t}$ and for the estimation we shall write the debt rate as time-varying.

B   Analytical Derivations

This appendix derives analytical expressions for asset prices and returns.

B.1   Asset Price

From the borrowers' optimality condition on risky assets:

$$q_t = \beta\mathbb{E}_t\left\{\frac{u_c(c_{t+1})}{u_c(c_t)}m_{t+1}(q_{t+1} + d_{t+1})\right\} + \phi\mu_t' q_t \tag{43}$$

$$= \beta\mathbb{E}_t\{\Lambda_{t,t+1}(d_{t+1} + q_{t+1})\} + \phi\mu_t' q_t$$

using the definitions for $\Lambda_{t,t+1} = \beta\frac{u_c(c_{t+1})}{u_c(c_t)}m_{t+1}$ and $\mu_t' = \frac{\mu_t}{u_c(c_t)}$. Then denoting $K_{t,t+1} = \frac{\Lambda_{t,t+1}}{1-\phi\mu_t'}$, we derive the following expression for the asset price:

$$q_t = \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + q_{t+1})\} \tag{44}$$

Proceeding by forward recursion:

$$q_t = \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}(d_{t+2} + q_{t+2}))\} \tag{45}$$

$$= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2})\} + \mathbb{E}_t\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}(d_{t+3} + q_{t+3})\}$$

$$= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2} + K_{t+1,t+2}K_{t+2,t+3}d_{t+3})\}+$$

$$+ \mathbb{E}_t\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}(d_{t+4} + q_{t+4})\}$$

$$= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2}+$$

$$+ K_{t+1,t+2}K_{t+2,t+3}d_{t+3} + K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}d_{t+4})\}+$$

$$+ \mathbb{E}_t\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}q_{t+4}\}$$

At the final recursion step, the solution for the asset price:

$$q_t = \mathbb{E}_t\left\{\sum_{i=1}^{T} d_{t+i}\prod_{j=1}^{i} K_{t+j-1,t+j}\right\} + \mathbb{E}_t\left\{\prod_{i=0}^{T} K_{t+i,t+i+1}q_{t+T}\right\} \tag{46}$$

Taking the limit for $T \to \infty$ of the above condition delivers equation (26).

B.2 The Risk Premium

Expanding the borrower's FOC for the risky asset and plugging in it the derivation for $\mathbb{E}_t\{\Lambda_{t,t+1}\}$ and the definition $R_{t+1}^s = \frac{q_{t+1}+d_{t+1}}{q_t}$ we get:

$$1 = \mathbb{E}_t\{\Lambda_{t,t+1}\frac{q_{t+1}+d_{t+1}}{q_t}\} + \phi\mu_t' \tag{47}$$

$$= \mathbb{E}_t\{\Lambda_{t,t+1}\}\mathbb{E}_t\left\{\frac{q_{t+1}+d_{t+1}}{q_t}\right\} + \mathrm{Cov}\left(\Lambda_{t,t+1}, \frac{q_{t+1}+d_{t+1}}{q_t}\right) + \phi\mu_t'$$

$$= \left(\frac{1-\mu_t'}{R}\right)\mathbb{E}_t\{R_{t+1}^s\} + \mathrm{Cov}(\Lambda_{t,t+1}, R_{t+1}^s) + \phi\mu_t'$$

The return on risky assets is obtained:

$$\mathbb{E}_t\{R_{t+1}^s\} = \frac{R(1 - cov(\Lambda_{t,t+1}, R_{t+1}^s) - \phi\mu_t')}{1-\mu_t'} \tag{48}$$

Dividing by the risk-free return rate, the premium between the return on the risky asset and the risk-free rate can be derived:

$$\Psi_t = \frac{1 - cov(\Lambda_{t,t+1}, R_{t+1}^s) - \phi\mu_t'}{1-\mu_t'}. \tag{49}$$

B.3 The Sharpe Ratio and the Hansen and Jagannathan (1991) Bounds

Writing down the two borrowers' optimal conditions for the risk-free and risky assets, respectively:

$$1 = \mathbb{E}_t\{\Lambda_{t,t+1}R\} + \mu_t' \tag{50}$$

$$1 = \mathbb{E}_t\{\Lambda_{t,t+1}R_{t+1}^s\} + \phi\mu_t' \tag{51}$$

where $\mu_t' = \frac{\mu_t}{u_c(c_t)}$, $\Lambda_{t,t+1} = \beta\frac{u_c(c_{t+1})}{u_c(c_t)}m_{t+1}$ and $R_{t+1}^s = \frac{q_{t+1}+d_{t+1}}{q_t}$. In order to derive the excess return between the risky asset and the risk-free asset, we subtract (50) from (51), obtaining:

$$0 = \mathbb{E}_t\{\Lambda_{t,t+1}(R_{t+1}^s - R)\} + \mu_t'(\phi - 1). \tag{52}$$

Then, we define the excess return as $z_{t+1} = R_{t+1}^s - R$. Assuming a linear general form for the stochastic discount factor $\Lambda_{t.,t+1}$:

$$\Lambda_{t.,t+1}^* = \bar{\Lambda}^* + \tilde{\beta^m}(z_{t+1} - E_t z_{t+1}) \tag{53}$$

it must satisfy the following condition:

$$0 = \mathbb{E}_t\{\Lambda_{t,t+1}^* z_{t+1}\} + \mu_t'(\phi - 1)\}, \tag{54}$$

which, once expanded, gives:

$$0 = \mathbb{E}_t\{\Lambda^*_{t,t+1}\}\mathbb{E}_t\{z_{t+1}\} + cov(\Lambda^*_{t,t+1}, z_{t+1}) + \mu'_t(\phi - 1) \tag{55}$$
$$= \mathbb{E}_t\{\Lambda^*_{t,t+1}\}\mathbb{E}_t\{z_{t+1}\} + \mathbb{E}_t\{(z_{t+1} - \bar{z})(\Lambda^*_{t,t+1} - \bar{\Lambda}^*)\} + \mu'_t(\phi - 1)$$
$$= \mathbb{E}_t\{\Lambda^*_{t,t+1}\}\mathbb{E}_t\{z_{t+1}\} + \mathbb{E}_t\{(z_{t+1} - \bar{z})(z_{t+1} - \bar{z})\tilde{\beta^m}\} + \mu'_t(\phi - 1)$$
$$= \mathbb{E}_t\{\Lambda^*_{t,t+1}\}\mathbb{E}_t\{z_{t+1}\} + \sigma_z^2\tilde{\beta^m} + \mu'_t(\phi - 1).$$

Hence:

$$\tilde{\beta^m} = -(\sigma_z^2)^{-1}\mathbb{E}_t\{\Lambda^*_{t,t+1}\}\mathbb{E}_t\{z_{t+1}\} - (\sigma_z^2)^{-1}\mu'_t(\phi - 1) \tag{56}$$

The variance of the stochastic discount factor is then obtained:

$$Var(\Lambda^*_{t,t+1}) = Var((z_{t+1} - \mathbb{E}_t\{z_{t+1}\})'\tilde{\beta^m}) \tag{57}$$
$$= \tilde{\beta'^m}\sigma_z^2\tilde{\beta^m}$$
$$= (-(\sigma_z^2)^{-1}\bar{\Lambda}^*_t\mathbb{E}\{z_{t+1}\} - (\sigma_z^2)^{-1}\mu'_t(\phi - 1))'\sigma_z^2$$
$$(-(\sigma_z^2)^{-1}\bar{\Lambda}^*_t\mathbb{E}\{z_{t+1}\} - (\sigma_z^2)^{-1}\mu'_t(\phi - 1))$$
$$= (\sigma_z^2)^{-1}(\bar{\Lambda}^*)^2(\mathbb{E}_t\{z_{t+1}\})^2 +$$
$$+ 2\mu'_t(\phi - 1)((\sigma_z^2)^{-1}\bar{\Lambda}^*_t\mathbb{E}\{z_{t+1}\} + ((\sigma_z^2)^{-1}(\mu'_t)^2(\phi - 1)^2.$$

Hence:

$$\frac{\sigma^2_{\Lambda^*_t}}{\bar{\Lambda}^{*2}} = \frac{(\mathbb{E}_t\{z_{t+1}\})^2}{\sigma_z^2} + 2\mu'_t\frac{(\phi - 1)\mathbb{E}_t\{z_{t+1}\}}{\sigma_z^2} + \frac{\mu'^2_t}{\Lambda^{*2}}\frac{(\phi - 1)^2}{\sigma_z^2}. \tag{58}$$

The Sharpe Ratio (SR hereafter) on stock asset returns over bonds results to be:

$$SR = \frac{(\mathbb{E}_t\{z_{t+1}\})^2}{\sigma_z^2} = \frac{\sigma^2_{\Lambda^*_t}}{\bar{\Lambda}^{*2}} - 2\mu'_t\frac{(\phi - 1)\mathbb{E}_t\{z_{t+1}\}}{\sigma_z^2} - \frac{\mu'^2_t}{\Lambda^{*2}}\frac{(\phi - 1)^2}{\sigma_z^2} \tag{59}$$

Thus, the SR depends on the variance of the adjusted for distorted beliefs stochastic discount factor and on μ'_t.

C Numerical Method

Our numerical method extends the algorithm of Jeanne and Korinek (2010) to persistent shocks and state-contingent ambiguity attitudes. The method, following the endogenous grid points approach of Carroll (2006), performs backwards time iteration on the agent's optimality conditions. We derive the set of policy functions $\{c(b, s), b'(b, s), q(b, s), \mu(b, s), V(b, s)\}$ that solve

competitive equilibrium characterized by the system:

$$c(b,s)^{-\gamma} = \beta R \, \mathbb{E} \left\{ m(b',s')c(b',s'^{-\gamma}) \right\} + \mu(b,s) \tag{60}$$

$$q(b,s) = \beta \, \frac{\mathbb{E} \left\{ m(b',s')c(b',s'^{-\gamma}[q(b',s') + \alpha y']) \right\}}{c(b,s)^{-\gamma} - \phi \mu(b,s)} \tag{61}$$

$$\mu(b,s) \left[\frac{b'(b,s)}{R} + \phi q(b,s) \right] = 0 \tag{62}$$

$$c(b,s) + \frac{b'(b,s)}{R} = y + b \tag{63}$$

$$V(b,s) = \frac{c(b,s)^{1-\gamma} - 1}{1 - \gamma} + \frac{\beta}{\sigma} \ln \mathbb{E} \left\{ \exp\{\sigma V(b',y')\} \right\} \tag{64}$$

where $m(b,s)$ is the expectation distortion increment. The solution method works over the following steps:

1. We set a grid $G_b = \{b_1, b_2, \ldots, b_H\}$ for the next-period bond holding b'; and a grid $G_s = \{s_1, s_2, \ldots, s_N\}$ for the shock state space $s = \{y, \sigma\}$. The income process y, is discretized with Tauchen and Hussey (1991) method, while the grid for the *inverse* of the penalty parameter σ (recall that θ is the inverse of σ) follows a simple two-state rule:[27]

$$\sigma = \begin{cases} \sigma^+ & \text{if} \quad V < \mathbb{E}\{V\} \\ \sigma^- & \text{if} \quad V \geq \mathbb{E}\{V\} \end{cases} \tag{65}$$

2. In iteration step k, we start with a set of policy functions $c_k(b,s)$, $q_k(b,s)$, $\mu_k(b,s)$ and $V_k(b,s)$. For each $b' \in G_b$ and $s' \in G_s$:

   a) we derive the expectation distortion increment:

   $$m_k(b',s') = \frac{\exp\{\sigma V_k(b',s')\}}{\mathbb{E}\left[\exp\{\sigma V_k(b',s')\}\right]} \tag{66}$$

   and then, the distorted expectations in the Euler equation for bonds and for the risky assets (equations (1) and (2)).

   b) we solve the system of optimality conditions under the assumption that the collateral constraint is slack:

   $$\mu^u(b',s) = 0 \tag{67}$$

   As a result, $c^u(b',s)$, $q^u(b',s)$, $\mu^u(b',s)$, $V^u(b',s)$ and $b^u(b',s)$ are the policy functions for the unconstrained region;

   c) in the same way, we solve the system for the constrained region of the state space, where the following condition holds:

   $$q^c(b',s) = -\frac{b'/R}{\phi} \tag{68}$$

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[27] We use 800 grids point for bonds and 45 grid points for the exogenous shocks; we implement linea interpolation in order to approximate the policy functions outside the grids

$c^c(b', s)$, $q^c(b', s)$, $\mu^c(b', s)$, $V^c(b', s)$ and $b^c(b', s)$ are the respective policy functions.

d) we derive the next period bond holding threshold $\bar{b}'$ such that the borrowing constraint is marginally binding. For each $s \in G_s$ it satisfies the following condition:

$$\bar{b}'^c(\bar{b}', s) + \frac{\bar{b}'(s)}{R} = 0 \tag{69}$$

When this point is out of the grid we use linear interpolation. Given this value, we can derive for each policy function the frontier between the binding and non-binding region: $x^u(\bar{b}'^c(\bar{b}', s)$ for $x = \{c, b, q, \mu, V\}$.

3. In order to construct the step k+1 policy function, $x_{k+1}(b, s)$, we interpolate on the pairs $(x^c(b'^c(b', s))$ in the constraint region, and on the pairs $(x^u(b'^u(b', s))$ in the unconstrained region. As a result we find: $c_{k+1}(b, s)$, $q_{k+1}(b, s)$, $\mu_{k+1}(b, s)$ and $V_{k+1}(b, s)$

4. We evaluate convergence. If

$$\sup ||x_{k+1} - x_k|| < \epsilon \qquad \text{for} \qquad x = c, q, \mu, V \tag{70}$$

we find the competitive equilibrium. Otherwise, we set $x_k(b, s) = (1 - \delta)x_{k+1}(b, s) + \delta x_k(b, s)$ and continue the iterations from point 2. We use a value of δ close to 1.

D   Data Description for Empirical Moments

In this section we describe the data employed for the computation of the empirical moments used for model matching. We compute several moments for asset prices, returns and debt data. Data are from the US. The used sample spans 1980:Q1 to 2016:Q4, since this corresponds to the period of rapid debt growth The dataset is composed as follows: debt is given by private non-financial sector, by all sectors (BIS: http://www.bis.org/publ/qtrpdf/r_qt1403g.pdf); consumption is given by Personal Consumption Expenditure (NIPA Tables[28]), GDP (NIPA Tables); the risk-free rate is the 3month T-bill rate (CRSP Indices database[29]); risky returns are proxied by the $S\&P500$ equity return with dividends (Shiller Database[30]). All variables are deflated by CPI index. Note that HP-filtered series are computed as deviations from a long-term trend. Therefore, we work with a much larger smoothing parameter ($\lambda = 400,000$) than the one employed in the business cycle literature, to pick up the higher expected duration of the credit cycle (see http://www.bis.org/publ/bcbs187.pdf).
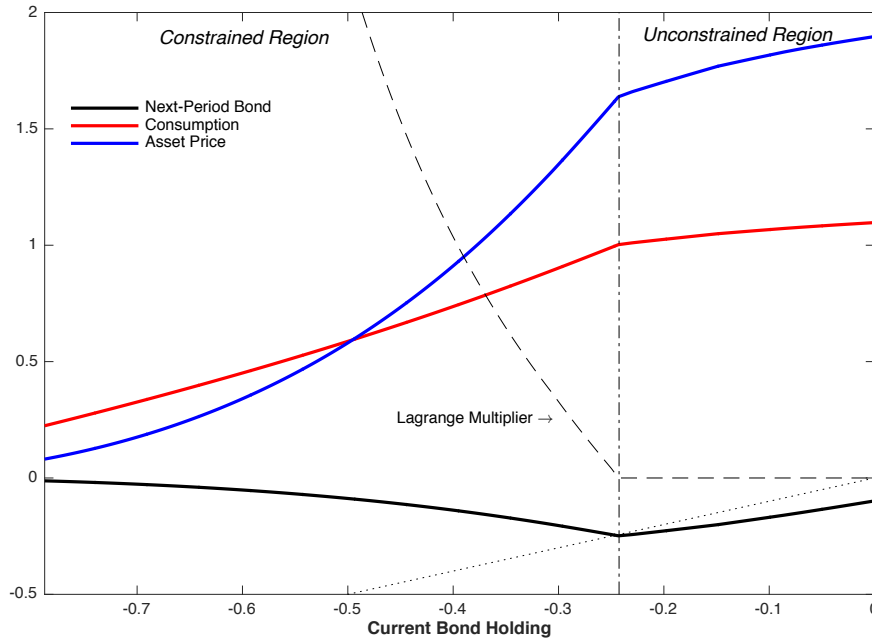
E   Policy Functions

Figure 6 shows the policy functions $c_t(b, y)$, $q_t(b, y)$, $b_{t+1}(b, y)$ and $\mu_t(b, y)$ for a medium income shock realization. It proves that our model, even with state contingent ambiguity attitudes,

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[28]See https://www.bea.gov/iTable/index_nipa.cfm.

[29]See http://www.crsp.com/products/research-products/crsp-historical-indexes.

[30]See http://www.econ.yale.edu/~shiller/data.htm

Figure 6: Policy Functions

is able to reproduce all the salient characteristics of the financial crises models (see Jeanne and Korinek (2010)). Indeed, in the binding region the next period bond holding is downward sloping and the policy functions for consumption and asset price display a higher inclination than in the unconstrained region. The latter feature implies that in the constrained region variables respond very strongly to changes in the current wealth, as the financial amplification theory states.

F Three Period Model

In this section we outline an extended version of the three period model with occasionally binding collateral constraints and with ambiguity attitudes. The goal is to show the combined effect of those two elements on debt growth. The economy we consider is populated by a continuum of agents, who live for three periods: $t = 0, 1, 2$. Preferences are given by the following specification:

$$U = u(c_0) + \mathbb{E}_S \left[\beta u(c_1) + \beta^2 u(c_2) \right] \tag{71}$$

where $u(c) = \frac{1}{2}[\bar{c} - c]^2$. In period 0 we can assume a linear utility function $u(c_0) = c_0$ in order to simplify the analysis. We also assume that $\beta R = 1$. The endowment structure is characterized as follows. Agents receive endowment income in period 1 and 2, but none in period 0. In period 1 the endowment is stochastic depending on the realization of the state $s \in S$. We assume that $S = \{s_1, s_2, ... s_N\}$ is a monotone increasing sequence. The realization of the endowment are affected monotonically from the realization of s, so that for example $y^{s_n} > y^{s_{n-1}}$. The probability that a state s occurs is given by π_s. Similarly to the main text we assume that the dividend is lead by the same source of volatility. This allows us to simplify the

state space. Therefore, in each period a fraction $(1 - \alpha)y_t$ is the labor income, and the fraction $d_t = \alpha y_t$ is the dividends' income. The budget constraints for each period reads as follows:

$$c_0 + q_0 x_0 + \frac{b_0}{R} = 0 \tag{72}$$

$$c_1^s + q_1^s x_1^s + \frac{b_1^s}{R} = (1 - \alpha)y_1^s + x_0(q_1^s + \alpha y_1^s) + b_0 \tag{73}$$

$$c_2^s = (1 - \alpha)y_2 + x_1 \alpha y_2 + b_1^s \tag{74}$$

Note that the sup-index s in period 1 indicates that uncertainty materializes in this period. We have assumed that $b_{-1} = b_2 = 0$, $x_{-1} = x_2 = 0$, $q_2 = 0$ and $d_{-1} = 0$. In period 1 the collateral constraint limits the amount of debt:

$$-\frac{b_1^s}{R} \leq \phi q_1^s x_1^s \tag{75}$$

The agents expectation formation process is derived as in the main text. Since uncertainty refers to period 1 income, agents form expectation in period 0. Their optimal likelihood ratio in period 0 is given by:

$$m_1^s = \frac{\exp\{\sigma_0 V_1^s\}}{\mathbb{E}_0\{\exp\{\sigma_0 V_1^s\}\}} \tag{76}$$

where the value function recursion is defined as following[31]: $V_1^s = u(c_1^s) + \beta u(c_2^s)$. The relation that links the level of m_1^s to the state of the economy is:

$$\text{if} \qquad V_1^s < \mathbb{E}_0\{V_1^s\} \qquad \text{then} \qquad m_1^s > 1 \tag{77}$$

Given the above optimization problems the *decentralized equilibrium* is characterized as follows. The bonds' Euler equations between periods 0 and 1 and between periods 1 and 2, read as follows:

$$1 = \beta R \mathbb{E}_0\{m_1^s u_c(c_1^s)\} \tag{78}$$

$$u_c(c_1^s) = \beta R u_c(c_2^s) + \mu_1^s \tag{79}$$

The Euler conditions on the risky asset between periods 0 and 1 and between periods 1 and 2 read as follows:

$$q_0 = \beta \mathbb{E}_0\{m_1^s u_c(c_1^s)[q_1^s + \alpha y_1^s]\} \tag{80}$$

$$q_1^s = \beta \frac{u_c(c_2^s)\alpha y_2}{u_c(c_1^s) - \phi \mu_1^s} \tag{81}$$

The complementarity slackness condition is:

$$\mu_1^s \left[\frac{b_1^s}{R} + \phi q_1^s\right] = 0 \tag{82}$$

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[31]This simplified representation is obtained under the assumption that there is no uncertainty in period 2.

Finally, the decentralized equilibrium is closed with a condition on expectations, equation (7), and the following market clearing conditions:

$$c_0 + q_0 + \frac{b_0}{R} = 0 \tag{83}$$

$$c_1^s + \frac{b_1^s}{R} = y_1^s + b_0 \tag{84}$$

$$c_2^s = y_2 + b_1^s \tag{85}$$

where we have imposed the stock market clearing condition $x_t = 1$.

F.1 Time 1 Continuation Equilibrium

We now proceed to the model solution by backward induction. We start from period the last period and since there is no uncertainty between time 1 and time 2 we can solve for the two periods simultaneously. We start from characterizing the continuation value under the *unconstrained region*. The system of equilibrium conditions for the unconstrained region (the sup-index U will be used since now on to indicate the solution for this region) is (we can use $\beta = R^{-1}$ and $\mu_1 = 0$):

$$u_c(c_1^s) = u_c(c_2^s) \qquad c_1^s = c_2^s = c^{U,s} \tag{86}$$

$$q_1^s = \beta \frac{u_c(c_2^s)}{u_c(c_1^s)} \alpha y_2 \tag{87}$$

$$c_1^s + \frac{b_1^s}{R} = y_1^s + b_0 \tag{88}$$

$$c_2^s = y_2 + b_1^s \tag{89}$$

Given the above the consumption function depends on lifetime wealth and reads as follows:

$$c^{U,s} = \frac{1}{1+\beta} \left(y_1^s + b_0 + \frac{y_2}{R} \right) \tag{90}$$

Using the budget constraint and the consumption function one can derive the optimal level of debt:

$$b_1^U(s) = \frac{\beta}{1+\beta} \left(y_1(s) + b_0 - \frac{y_2}{R} \right) \tag{91}$$

Finally, the equilibrium asset price condition, which depends on the value of the dividend in the last period, reads as follows:

$$q_1 = \beta \alpha y_2 \tag{92}$$

In the *constrained region* ($\mu_t > 0$, the sup-index C is used since now onward to indicate

equilibrium values for this region)), the system of equilibrium conditions reads as follows:

$$\mu_1^s = u_c(c_1^s) - u_c(c_2^s) \qquad c_1^s < c_2^s \tag{93}$$

$$q_1^s = \beta \frac{u_c(c_2^s)}{u_c(c_1^s) - \phi\mu_1^s} \alpha y_2 \tag{94}$$

$$c_1^s + \frac{b_1^s}{R} = y_1^s + b_0 \tag{95}$$

$$c_2^s = y_2 + b_1^s \tag{96}$$

$$\frac{b_1^s}{R} = -\phi q_1^s \tag{97}$$

F.2 Time Zero Equilibrium

To characterize the time 0 equilibrium we first partition the state space into two blocks, S^C and S^U, where the constraint is binding and slack respectively. Assuming that the $u(c_0) = c_0$ we have:

$$1 = \sum_{s \in S^U} \pi_s m_1^{U,s} u_c^{U,s}(b_0; y_1, y_2)) + \sum_{s \in S^C} \pi_s m_1^{C,s} u_c^{C,s}(b_0; y_1, y_2)) \tag{98}$$

$$q_0 = \beta \left\{ \begin{array}{c} \sum_{s \in S^U} \pi_s m_1^{U,s} u_c^{U,s}(b_0; y_1, y_2))[q_1^U + y_1^s] \\ + \sum_{s \in S^C} \pi_s m_1^{C,s} u_c^{C,s}(b_0; y_1, y_2))[q_1^{C,s}(b_0; y_1, y_2) + y_1^s] \end{array} \right\} \tag{99}$$

$$c_0 = -\frac{b_0}{R} - q_0 \tag{100}$$

where $c_1^{i,s}, b_1^{i,s}, q_1^{i,s}$ are the solutions of the time 1 continuation equilibrium.

F.3 The Expectation Distortion under a Binomial State Space

Our goal is to assess the role of ambiguity attitudes on debt growth. To this purpose we shall derive a closed form solution for policy functions. To do that we assume a simple binomial structure for the state space. Hence we assume that the state space is comprised of two states, which we label high, with sup-index h, occurring with probability π, and low, with sup-index l, occurring with probability $(1 - \pi)$. The exogenous state space therefore reads as follows $S = \{h, l\}$. We assume that the in state h the income realization is high enough that the collateral constraint is slack. Similarly we assume that in state l, the income realization is low enough that the collateral constraint binds. Given this structure for the objective probability, the expectation distortions are given by:

$$m_1^s = \frac{\exp\{\sigma_0 V_1^s\}}{\pi \exp\{\sigma_0 V_1^h\} + (1 - \pi)\exp\{\sigma_0 V_1^l\}} \tag{101}$$

where the value function has the following form, $V_1^s = u(c_1^s) + \beta u(c_2^s)$. Given the assumptions on the state space, it follows that:

$$V_1^h > \mathbb{E}_0\{V_1^s\} \quad \text{and} \quad V_1^l < \mathbb{E}_0\{V_1^s\} \tag{102}$$

Equation (101) and jointly imply that, if $\theta_0 > 0$, hence $\sigma_0 = -\frac{1}{\theta_0} < 0$, the following holds:

$$\exp \sigma_0 V_1^h < \mathbb{E}_0 \{\exp \sigma_0 V_1^s|\} \quad \Rightarrow \quad m_1^h < 1 \tag{103}$$

$$\exp \sigma_0 V_1^l > \mathbb{E}_0 \{\exp \sigma_0 V_1^s\} \quad \Rightarrow \quad m_1^l > 1 \tag{104}$$

Intuitively the above implies that agents assign an higher subjective probability (with respect to the objective) to the bad history and a lower probability to the good history. We can call this behaviour *pessimism*. Similarly if $\theta_0 < 0$, then $\sigma_0 = -\frac{1}{\theta_0} > 0$, we have that:

$$\exp \sigma^0 V_1^h > \mathbb{E}_0 \{\exp \sigma_0 V_1^s\} \quad \Rightarrow \quad m_1^h > 1 \tag{105}$$

$$\exp \sigma_0 V_1^l < \mathbb{E}_0 \{\exp \sigma_0 V_1^s\} \quad \Rightarrow \quad m_1^l < 1 \tag{106}$$

Note that in this second case agents assign an higher subjective probability to the good history and a lower probability to the bad history, depicting borrowers' *optimistic* behaviour. We shall now solve the equilibrium and derive the implied debt policy functions under the above beliefs' structure. We start by characterizing the equilibrium at time zero, given by the optimal decisions (b_0, c_0, q_0). We also compare the two solutions to the case with rational expectations. The debt policy function is best characterized by the following relation:

$$b_0 = -R[c_0 + q_0] \tag{107}$$

Next to characterize the time 0 policy function for consumption we rely on the Euler equation between period 0 and period 1:

$$u_c(c_0) = \pi m_1^h u_c(c_1^h) + (1 - \pi) m_1^l u_c(c_1^l) \tag{108}$$

We can reformulate the above equation in terms of the subjective weights of the ambiguity averse agent:

$$u_c(c_0) = \psi^h u_c(c_1^h) + (1 - \pi) \psi^l u_c(c_1^l) \tag{109}$$

where $\psi^h = \pi m_1^h$ and $\psi^l = (1 - \pi) m_1^l$. Given the model structure (incomplete financial markets, hence lack of insurance to equalize consumption), the events structure and the condition on the collateral constraint, we can conclude that:

$$c_1^h > c_1^l \quad \Rightarrow \quad u_c(c_1^h) < u_c(c_1^l) \tag{110}$$

Next, recall that in the *optimism* case beliefs imply:

$$\psi^h = \pi m_1^h > \pi \tag{111}$$

$$\psi^l = (1 - \pi) m_1^l < (1 - \pi) \tag{112}$$

This implies that agents assign a higher weight, with respect to the RE case, to the component $u_c(c_1^h)$. Hence, the marginal utility of consumption in $t = 0$ is lower (than under rational

expectations) and the consumption is higher:

$$c_0^o > c_0^{RE} \tag{113}$$

where c_0^o indicates consumption under optimism behaviour, while c_0^{RE} indicates consumption under no ambiguity. Intuitively agents assign higher weight to good future states, hence they prefer to postpone consumption and to invest in the risky asset. This in turn will raise asset price, since the demand of asset has increased. As investment takes place through leverage, they will also leverage more. In the *pessimism* case the borrower assigns the following weights:

$$\psi^h = \pi m_1^h < \pi \tag{114}$$
$$\psi^l = (1-\pi)m_1^l > (1-\pi) \tag{115}$$

This implies:

$$c_0^u < c_0^{RE} \tag{116}$$

where c_0^u indicates consumption under pessimistic behaviour. In this case the agent expects more likely the bad state to take place in the future. The agent will then anticipate consumption and invest less in the risky asset. They will in turn leverage less. We can generalize this relation with the following condition:

$$c_0^o > c_0^{RE} > c_0^u \tag{117}$$

G   Intermediation Channel

In this section we provide micro-foundations for a delegated monitoring problem in which the collateral constraint emerges as resulting from an incentive-compatible debt contract enforced through a bank. The micro-foundations follows Bianchi and Mendoza (2015). Debt contract are signed by a bank that must enforce debtor incentives. Between periods borrowers can divert revenues for an amount $\tilde{d}$. At the end of the period the diversion is no longer possible and payment is enforced. Banks can monitor financial diversion due to special relationship lending abilities[32]. If the bank detects the diversion asset can be seized up to a percentage ϕ. As common in dynamic economies we assume that the contract is done under no memory, so that in the next periods borrowers can re-enter debt agreement even if they defaulted in the previous period. This assumption allows us to preserve the Markov structure of the contracting/intermediation problem.

We shall show that the collateral constraint can emerge as resulting from an incentive compatibility constraint imposed by the bank through the debt design. Specifically the collateral constraint can be derived as an implication of incentive-compatibility constraints on borrowers if limited enforcement prevents banks from redeploying more than a fraction ϕ of the value of the assets owned by a defaulting borrower. Define V^R and V^D respectively the value of repayment

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[32]We assume zero monitoring costs for simplicity. Extending it to the case with positive monitoring costs is rather straightforward.

and default and define as V the continuation value.

If the borrower defaults the diverted resources enter his budget constraint and the recursive problem reads as follows (for notational convenience we skip the beliefs constraints for the purpose of this derivation):

$$V^D(b, x, S) = \max_{c, x', b'} \{ u(c) + \beta \mathbb{E}_S + \tag{118}$$

$$+ \lambda \left[y + q(S)(x + \alpha y) + \tilde{d} + b - q(S)x' - c - \frac{b'}{R} \right] +$$

$$+ \mu \left[\phi q(S)x' + \frac{b'}{R} \right]$$

On the other side if the borrower repays his value function reads as:

$$V^D(b, x, S) = \max_{c, x', b'} \{ u(c) + \beta \mathbb{E}_S + \tag{119}$$

$$+ \lambda \left[y + q(S)(x + \alpha y) + b - q(S)x' - c - \frac{b'}{R} \right] +$$

$$+ \mu \left[\phi q(S)x' + \frac{b'}{R} \right]$$

The comparison of the two easily shows that the households repay if and only if $\tilde{d}' < \phi q(S)x'$.