On the Evolutionary Fitness of Rationality

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Abstract

This work analyses the interaction of perfectly rational agents in a market with coexisting boundedly rational traders. Whether an individual agent is perfectly rational or boundedly rational is determined endogenously depending on each types market performance. Perfect rationality implies full knowledge of the model including the non-linear switching process itself. Policy function iteration is used to find a recursive minimal state variable solution of the highly nonlinear system and I show that this solution is not necessarily bounded. Depending on the parameterization, agents' interaction can trigger complicated endogenous fluctuations that are well captured by the solution algorithm. In such financial market setup rational agents might adapt sentiment beliefs and so fail to mitigate speculative behavior, and boundedly rational agents are not necessarily driven out of the market. While up to a certain point the presence of fully rational agents tends to have stabilizing effects it may later amplify endogenous fluctuations.

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1 Introduction

> *"If any group of traders was consistently better than average in forecasting stock prices, they would accumulate wealth and give their forecasts greater and greater weight. In this process, they would bring the present price closer to the true value."*
>
> — Cootner, 1964

The above quote encapsulates one of the cornerstones of economic theory: the Rational Expectations Hypothesis holds that in an efficient market, agents that do not act rationally will be outperformed by those who move wisely and well-informed. Agents that underperform for a longer period are then driven out of the market, leaving only rational agents behind (Friedman, 1953). But is also postulates that rational agents will foresee the fundamental price, instead of the *true* price, which might be affected by sentiment traders. *Perfect Rationality* however implies not only the knowledge on the economic fundamentals, but also on the market itself and its participants.

To study the interaction of perfectly rational and sentiment traders in a market with positive feedback and to reassess the Rational Expectations Hypothesis I extend the Brock-Hommes Heterogeneous Agent Switching Model (Brock and Hommes, 1998, referred to as BH) to incorporate fully rational agents. These agents are *hyper-rational* as they do not only know the market environment, but also they are aware of the behavior of any other agent in the market, their expectations on the future, their relative number today and, in expectations, for all periods to come. Put differently, they are rational in the classical sense and they are aware of the presence of non-rational agents – their expectations are model consistent and do not contain any misspecification.

Using the BH98 model this work focuses on two questions. First, I investigate whether rational agents are actually stabilizing the market. This problem is particularly interesting because there are two contradicting intuitions. On the one hand, the presence of rational agents could indeed have a stabilizing effect. These agents foresee other agents behavior perfectly and might be able to outsmart them. Technically, the rational expectations solution is a fixed point in the law of motion: the expected value of the price next period determines the outcome at the present period while it is also path-dependent upon the latter. On the other hand, the presence of rational agents is likely to be destabilizing. Since rational agents know all other beliefs, they anticipate the behavior of boundedly rational agents, trade accordingly and thereby positively reinforce the behavior and beliefs of bounded rational agents. In this case the Rational Expectation Hypothesis could never hold, because sentiment beliefs are ex-ante true through their reinforcement by rational agents. Secondly and much related, I ask whether a reasonably large fraction of boundedly rational agents can survive in the medium term and long term.

The concept of Rational Expectations has first been introduced by Muth (1961) and in particular gained popularity through the work of Lucas (1972). Although the leading paradigm today, it also received early critique as most prominently by Simon (1955), and counter-critique e.g. by (Sims, 1980, "the wilderness of bounded rationality"). Kirman (1992) is a more contemporary response to the *rational expectations revolution*, which has redrawn attention on the matter. As of yet, this debate is unsettled. Although many major economists agree that the assumptions underlying the Rational Expectations framework might be too demanding, as of yet a clear and commonly accepted alternative has not emerged. For a deeper discussion on the concept of bounded rationality see f.i. Conlisk (1996).

With closer ties to the scope of my work here, the idea of "a superior analyst" that is capable of outsmarting less sophisticated agents has also been discussed in the seminal article of Fama (1965). In his view a sufficiently large fraction of smart agents would be able to prevent bubbles. In contrast, De Long et al. (1990) support the idea that rational traders might amplify price swings that are induced by noise traders, if rational agents can anticipate other agents trading behavior and can act ahead of time. Other than in their model, I explicitly consider the behavior of boundedly rational agents and allow for the fraction of rational agents to be endogenous. Hens and Schenk-Hoppé (2005); Amir et al. (2005) present evidence that non-CAPM trading strategies might under some circumstances be the only evolutionary stable strategy, whereas Evstigneev et al. (2002) provide evidence that such strategies might be able to consume the whole market. Work of Blume and Easley (2010) investigate similar questions in a different setup. They empathise the fact that the market selection hypothesis fails when markets are incomplete and discount factors are heterogeneous.

The state of the art of the research conducted on bounded rationality in the BH-tradition can be found in Hommes (2013). Using not only behavioral models but also laboratory experiments, this branch of research emphasises that the stability of a system crucially depends on the type and degree of expectations feedback. Negative feedback loops, although not under every circumstances, tend to be relatively stable since expectations are not self-enforcing. In the case of a positive feedback loop, stability depends crucially on the magnitude of the eigenvalues. In this work I focus on financial markets since in this area the positive feedback is apparent. I do not discuss the case of commodity markets with negative feedback here since this problem normally embeds an explicit dynamic system also under rational expectations and is well known to the literature as the so-called *hog-cycle model* (Brock and Hommes, 1997).

Fully rational agents in a financial market content have, in this stand of research, only been studied in Brock et al. (2009). The authors focus on the question whether market completeness can improve price stability while also touching on the question of whether rational agents stabilize the market. Rationality is then derived from the perfect foresight argument while ruling out bubble solutions, i.e. reducing the degrees of freedom of the perfect foresight solution by one. Instead of finding an explicit representation of the implied actual law of motion, they provide analytical results and conclude that whether rational agents can stabilize the market or not highly depends on the degree and composition of boundedly rational fraction of agents in the market.

The key methodological contribution of this work is an iterative numerical method to solve for an explicit representation of the rational expectations equilibrium. To my best knowledge, such methods have not been applied to the form of highly nonlinear models with endogenous fluctuations as the one considered here. Methods of this type are well known to dynamic economic theory and the field of recursive macroeconomics (e.g. Ljungqvist and Sargent, 2012). See Judd (1998) and Miranda and Fackler (2004) for comprehensive, general surveys of numerical methods in economics. The iterative method considered here allows to explicitly account for fully rational agents in an Heterogeneous Agent Switching Model (HAM), whereas the previous literature focussed on concepts that embed a closed form solution for the law of motion. As such, the model was mainly studied with fundamentalist traders or approximating rational agents by using the concept of the perfect foresight path. These concepts do not require to solve for the rational expectations solution, but, as I explain further below may not conceptualize the rational expectations solution correctly or, as for fundamentalists, not be well suited to answer the question that drives this work.

The rest of this work is structured as follows. In Section 2 I briefly introduce the model and sketch the numerical solution method in Section 3. In Section 4 I present and discuss the simulation results. Section 5 concludes.

2 Model

This section attempts to stay as close as possible to the original model of Brock and Hommes (1998) as this model is well established in the literature of bounded rationality and nonlinear economic dynamics and provides a well-known reference point. I hence follow their derivation of the model closely.

Accordingly, let us consider a stylized asset market with a continuum of agents that are neither constrained in borrowing nor in short-selling. Furthermore, each trader is a myopic mean variance maximizer, which implies that trader i's demand $z_{i,t}$ for the risky asset is a linear function of his beliefs $x^e_{i,t+1}$ about the price in $t+1$ as well as todays price. If x_t is defined to be the percentage deviation of the price for the financial asset at time t from its fundamental value, market clearing reads then as a no-arbitrage condition of the form

$$Rx_t = \int x^e_{i,t+1} di,$$

where R stands for the (time-invariant) discount rate.This equation will also be called the *law of motion* (LOM) of the model.

For simplicity this work is restricted to a family of models with a maximum of three types of agents of which two types are symmetrical. This is sufficient for the purpose of this work and covers a wide range of possible dynamics while still allowing for a parsimonious model. The solution concept for the Rational Expectations path could however easily be adapted to more complicated models with more types.

In particular agents are either sentiment traders, i.e. optimistic or pessimistic about the near future, or perfectly rational and then $x^e_{i,t+1} \in \{x^{e+}_{t+1}, x^{e-}_{t+1}, E_t x_{t+1}\}$. The beliefs $E_t x_{t+1}$ of this third group of perfectly rational agents are formed based on the information available at t and the complete knowledge of the model. I present and discuss the solution concept for the expectations of these agents in the next section.[1] Let me formalize the predictors of sentiment traders by

$$x^{e+}_{t+1} = +\beta \qquad \text{and} \qquad x^{e-}_{t+1} = -\beta,$$

where the degree of sentiment bias is denoted by β. Market clearing is then given by

$$Rx_t = (1 - n_{+,t} - n_{-,t})E_t x_{t+1} + (n_{+,t} - n_{-,t})\beta, \tag{1}$$

[1] In the rest of this work I am using the terms *rational* and *rational expectations* interchangeably. To be precise, agents could form perfectly rational expectations but act boundedly rational or even irrationally [whith respect to utility/profit maximisation] . In models of the BH-type, in fact all agents *act* fully rationally *given their beliefs*. Their choice of predictors however might not be completely rational. This, again, is to remain consistent with the majority of the literature.

where $n_{+,t}$ denotes the fraction of optimists and, likewise, $n_{-,t}$ the fraction of pessimists. Further following Brock and Hommes (1998), these fractions are updated according to the *performance measure* $\pi_{i,t}$ for each predictor. As such, *realized profits* is a natural candidate and I define:

$$\pi_{i,t} = (x_t - Rx_{t-1})(x^e_{i,t} - Rx_{t-1}) - \mathbb{1}_{RE}\kappa, \tag{2}$$

where κ is the cost for obtaining the rational expectations solution and $\mathbb{1}_{RE}$ an indicator function that equals one if the agent is rational. All other predictors are costless. The first part of the first term at the RHS of (2) denotes the actual resale value of the asset minus the opportunity costs for financing the purchase in the previous period. The second part represents agent *i*s demand, resulting from the mean-variance maximization given the agent's past belief about the price. The choice of the performance measure is an essential ingredient of the model. It determines the properties of the dynamic system. Realized profits from trading qualify in several ways for this model. Instead of receiving a high pay-off for an accurate estimate of the price, in order to receive a positive profit it is sufficient to have made a correct choice on whether to go short or long. Likewise, a trader A that has a strong positive belief about next periods prices will invest more money in the asset than another trader B with a relatively lower positive forecast. Even if B's forecast was perfectly correct, trader A will still make higher profits since he invested more. This feature, i.e. that profits are non-proportional to forecasting errors, is unique to financial markets and captured by Equation (2).

The probability that an agent is of type $i \in \{+, -, RE\}$ is determined by a *multinomial discrete choice model* depending on the past performance of the predictor:

$$n_{i,t} = \frac{e^{\delta \pi_{i,t-1}}}{\sum_{j \in N} e^{\delta \pi_{j,t-1}}}. \tag{3}$$

If a predictor is relatively more successful than others, it is more likely to be chosen, hence the fraction of agents using this predictor increases. δ is called the *intensity of choice* which governs the speed of switching between predictors. If $\delta \to \infty$, all agents will immediately switch to the most successful predictor. This completes the full specification of the formal model.

3 Numerical solution

We are looking for a representation such that at any point t the state of the system can be calculated given only the *relevant* past states as implied by the law of motion. The model presented in the previous section does not allow for such a solution in closed form. In the literature on nonlinear economic dynamics it is sometimes argued that, in the absence of stochastic disturbances, the Rational Expectation Path (REP) can be found by iterating the law of motion backwards.

Let me call this concept to be the *backwards consistent solution.* To remain general, consider a dynamic forward looking model h that represents the state at time t by

$$y_t = h\left(E_t[y_{t+1}], y_{t-1}, \cdots, y_{t-k}\right), \tag{4}$$

where h is some mapping from $\mathbb{R}^{k+1}$ to $\mathbb{R}$ and $(\cdot)^e$ is an expectation operator that is yet to be defined.

Given a dynamic forward looking model h and a sufficiently long history, each z_t in $\{z_t\}_0^\infty$ satisfies

$$z_t = \{y_{t+1} : y_t = h(y_{t+1}, y_{t-1}, \cdots, y_{t-k})|y_t, y_{t-1}, \ldots, y_{t-k}\},$$

i.e. it is imposed that $\{y_t\} = z_{t-1}$ and every y_{t+1} is chosen given y_t. The backwards consistent solution can then said to be **ex post model consistent**.

As I show here, the the backwards consistent solution does in general not coincide with the REP. In particular, the backwards consistent solution embeds one additional degree of freedom and includes the REP as a special case. The expectation $E_t y_{t+1}$ is by itself not a variable but a function of the control variable y_t. This implies that both objects are defined only by the history of y starting with y_{t-1}. h hence does have k degrees of freedom. To clarify, given a sufficiently long history $\{y_{t-1}, y_{t-2}, \cdots\}$, the REP $\{y_t\}_k^\infty$ is defined to satisfy (4) in each period $t > k$ and $E_t y_{t+1}$ equals the expected value of $y_{t+1} = h(\cdot, y_t, \cdots)$ based on the information set implied by the history at time t. The Rational Expectations solution associated with the REP can then said to be **ex ante model consistent**.

Put differently, $E_t y_{t+1} = E[y_{t+1}|y_t, \cdots, y_{t-k}]$, and each y_t on the REP with $t \in \mathbb{R}^+$ must be consistent with $E_t y_{t+1}$. Hence, $E_t y_{t+1}$ represents a fixed point. In fact, in the absence of stochastic shocks, it is deterministic and coincides with the actual y_{t+1}. Note that this implies that each y_t on the REP is a mapping $y_t : \mathbb{R}^k \to \mathbb{R}$, meaning that only the history of length k is necessary to compute y_t.

In contrast, the backwards consistent solution implies that z_t is a mapping $z_t : \mathbb{R}^{k+1} \to \mathbb{R}$. Now a history of length $k + 1$ is necessary to find z_t which is a larger set than what actually occurs in h. In Appendix Appendix A I deepen the intuition that the backwards consistent solution coincides with the Rational Expectations Path (REP) only if *all* initial conditions of the backwards first lie on the REP. It is easy to see that this equivalence condition is not satisfied in general. Since it can only be guaranteed once the REP is known, the Perfect Foresight Solution is not a particularly helpful tool when solving dynamic models with rational expectations, independently of whether they are deterministic or stochastic.

The numerical procedure used here to solve for the REP is closely related to the method of Policy Function iteration with Time Iteration (see e.g. Coleman, 1990; Judd, 1998; Ljungqvist and Sargent, 2012) which is well known to the literature of e.g. dynamic games or nonlinear macroeconomics. The basic intuition goes as follows: at any point in time $t + s > t$ the future is unknown and hence can not be used to find the systems' state at t. If we however would have a solution to the system, it could be used as well to infer on any future state x_{t+s}. The existence of such representation implies a solution for REP. The method first assumes existence of a solution and then verifies by finding its exact representation. This is in fact equivalent to a rational agent facing a decision that will affect the future, while the future is relevant to make the decision - the so-called *fixed point argument* that is implied by $E_t y_{t+1}$ being a function of y_t.

Plugging (2) in (3) and inserting the result into (1), the model's state at t can be expressed as a (known, nonlinear) function f that depends on the rational expectation of next periods price, as well on the past values:

$$f : (E_t x_{t+1}, x_{t-1}, x_{t-2}) \to \mathbb{R}.$$

Note that in the absence of shocks it also holds that $E_t x_{t+1} = x_{t+1}$, which however is not a necessary condition for this method to work. Being able to solve for the Rational Expectations Path of this system implies that there exists a recursive representation g that is a (unknown) function of only the history in f:[2]

$$g : (x_{t-1}, x_{t-2}) \to \mathbb{R}. \tag{5}$$

g can be found by inserting it into f. We know:

$$\begin{aligned} x_t &= f(x_{t+1}, x_{t-1}, x_{t-2}) \\ &= g(x_{t-1}, x_{t-2}) \end{aligned}$$

and

$$\begin{aligned} x_{t+1} &= g(x_t, x_{t-1}) \\ &= g(g(x_{t-1}, x_{t-2}), x_{t-1}). \end{aligned}$$

Then our problem boils down to finding a function g that satisfies

$$g(x_{t-1}, x_{t-2}) = f(g(g(x_{t-1}, x_{t-2}), x_{t-1}), x_{t-1}, x_{t-2}), \tag{6}$$

which can be done numerically. For this purpose, let us define $\mathbf{x} = \{X_1, X_2, \ldots, X_M\}$ a vector of M grid points on which g shall be defined. g then resides on $\mathbb{R}^{M\times M}$, which in the two-dimensional case is a matrix. Given an initial guess g_0 we can iterate (6)

$$g_{k+1}(\mathbf{x}, \mathbf{x}') = f(g_k(g_k(\mathbf{x}, \mathbf{x}'), \mathbf{x}), \mathbf{x}, \mathbf{x}').$$

If g exists, $\|g_{k+1} - g_k\|$ converges to zero when k goes to infinity. The iteration halts once a $\|g_{k+1} - g_k\| < \epsilon$ for some predefined very small ϵ is reached.[3]

4 Results

In this section five different types of experiments are presented. First, I explore the potential dynamics of the system by varying the behavioral parameters δ and β. Second, to study the effect of rational traders in the market I compare the dynamics of different values of costs for rational expectations κ. Then I compare these results with a model including fundamentalists instead of rational agents. I furthermore revisit the BH two-trader type model. Lastly, to identify further mechanisms and to provide robustness I compare these results to a model with a risk-adjusted fitness measure.

[2]Note that in a linear framework this would imply the Transversallity Condition since it is a sufficient condition for the existence of a recursive solution. This can be seen e.g. by using Eigenvalue-Eigenvector Decomposition. Numerical procedures of this type generally do not require the Tranversallity Condition for stationarity.

[3]In the Appendix I deepen further on conditions under which the iteration might not converge.

R	δ	β	κ
0.99^{-1}	1	1	0

Table 1: Benchmark parametrisation

As the benchmark the parameters given in Table 1 are used. The two behavioral parameters δ and β are normalized to unity for simplicity and costs for rational expectations are set to zero.

4.1 Endogenous dynamics

In order to asses the general dynamics I am looking at the dynamics under different values for the behavioral parameters β and δ. This also sheds light on the question whether boundedly rational agents might get driven out of the market, although the discrete choice model does not allow for a zero-fraction of sentiment traders. This however can also be seen as a realistic feature since in real markets there will always be new market entrants that might as well be boundedly rational. The same strategy is chosen by Brock et al. (2009).

In Figure 1 the long-run dynamics of prices and fractions of each type of agents are shown as a function of the intensity of choice δ. For low values of this parameter, the simulations suggest that the steady state is stable and unique. As δ increases, a limit cycle emerges after what appears to be a Hopf-Bifurcation at $\delta \approx 1.2$, which is a typical characteristic of a 3-trader-type-model. The amplitude of these cycles increases in δ.

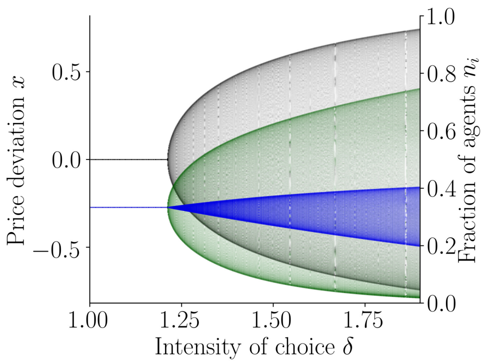

Figure 1: Bifurcations w.r.t. δ. In blue (green) and on the right axes the dynamics of the fraction of rational (sentiment) traders.

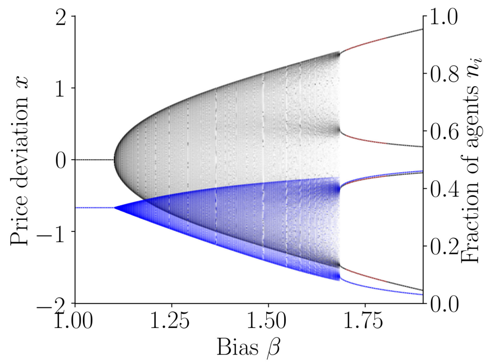

Figure 2: Bifurcations w.r.t. β. In blue and on the right axes the dynamics of the fraction of rational traders.

An increase in the intensity of choice does also lead to an increase in the variation of the amount of rational agents, that is depicted in the blue region, while the green region shows fluctuations in the amount of optimistic agents.[4] Why does the number of sentiment agents vary significantly stronger than the number of rational agents? In

[4]Note that, due to the symmetric construction of optimist and pessimist traders, the dynamics of optimist and pessimist traders are in fact symmetric.

a bearish period the number of optimist traders will fall because the belief that prices are high is not profitable. Hence, in this period most boundedly rational agents will be pessimists. While rational agents always forecast the price correct, their profit and hence their selection only depends on $x_t - x_{t-1}$. Pessimist traders however will gain higher profits since they have overestimated the change in price and hence decreased their short position more strongly. This relatively higher profit, in extreme periods, does also motivate some of the rational agents to turn pessimistic. Because the intensity of choice governs how strongly agents react to incentives by higher profits, the overall fluctuations in optimist/pessimist traders increases with δ. The dynamics for $\delta \to \infty$ can be seen in Figure 5, which turns out to be a 4-cycle similar to the original BH-model.

Figure 2 shows the dynamics with respect to the degree of optimism and pessimism β. While these dynamics are qualitatively similar to those in Figure 1, the amplitude increases faster. The more biased agents are, the stronger they influence the price in troughs and peaks. Up from a value of $\beta \approx 1.7$ the system converges as well to a stable 4-cycle. This can be seen as further evidence against the aforementioned hypothesis: starting from a small fraction of boundedly rational agents, if their beliefs are strong enough they are again able to influence the market in such a way that rational agents will partially adopt and reinforce their beliefs in their forecast.

This already reveals, given the model assumptions here, that the hypothesis that boundedly rational agents are driven out of the market in the long run might be controvertible. If agents tend to switch to the more successful strategy more quickly, there is always a fraction of boundedly rational agents that is sufficiently successful to gather at least a share of the market. Any rational agent then adjusts his belief accordingly and by doing so amplifies the boundedly rational traders' beliefs. Although this model does not explicitly allow for agent types to be driven out of the market, the fact that the agents' fraction fluctuates around one-third suggests a falsification of the REH.

4.2 The consequences of rationality

Let us now have a look at Figures 3 and 4 where the costs for the rational expectations operator are plotted against the x-axis. This allows to study the effect rational agents have, given that an increase in costs is associated with a decrease of the number of rational agents.

The effect of an increase in costs for rationality is twofold. As suggested by Figure 2, the steady state for $\kappa = 0$ is stable and unique for $\beta = 1.05$. Taking this as a starting point, in Figure 3 I let $\beta = 1.05$ and increase κ. The blue area confirms that the fraction of rational agents decreases with the costs for rationality while simultaneously the fraction of sentiment traders increases. Once this fraction is large enough, again a Hopf-Bifurcation occurs and endogenous speculative dynamics arise. A further decrease in the number of rational agents does not seem to have large impact on the price dynamics, and in fact the dynamics will be identical for any value of $\kappa > 4$. In this setup the presence of a significant number of rational agents does add stability and inhibits or at least mitigates the degree of endogenous fluctuations. This experiment hence supports the hypothesis that a reasonably large fraction of rational agents can indeed stabilize the market and bring prices closer to fundamentals.

The implications of Figure 4 however draw a rather ambiguous picture. Here sentiment traders are biased more strongly which, as suggested by Figure 2, leads to a 4-cycle even if rationality is costless. Dynamics then become quasi-periodic for values of

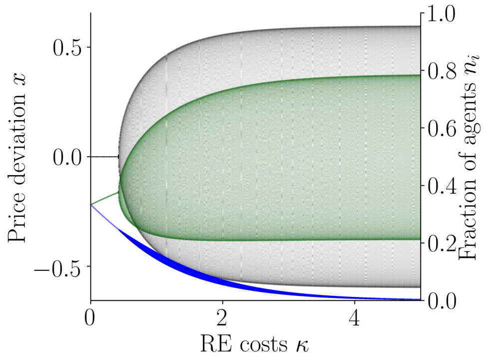

Figure 3: Bifurcations w.r.t. κ. $\beta = 1.05$. Blue and green are the fractions of rational and positive biased agents.

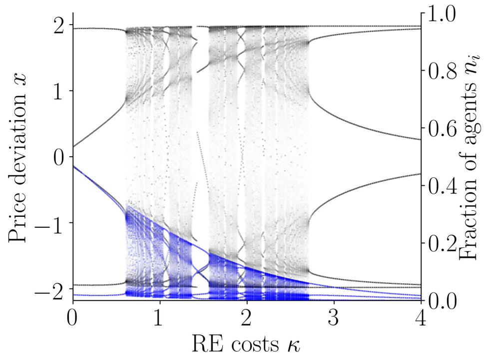

Figure 4: Bifurcations w.r.t. κ. $\beta = 2$. In blue dynamics of the fractions of rational agents.

κ smaller than approximately 2.7 are suggested by the diagram for higher κ. A further decrease in the number of rational agents then leads to a stable 6-cycle as κ goes to infinity. This result can be interpreted such, that once – for one reason or another – endogenous fluctuations arise, a variation in the fraction of rational agents can not add further stability (and costs can not be set ¡ 0). This is well in line with the previous argument that either a high intensity of choice or stronger biases after some point affect the market such, that rational agents to some extend have to adapt the boundedly rational beliefs.

4.3 Comparing rational and fundamentalist traders

Fundamentalists traders are agents who believe that the price will always return to its fundamental value, which is here given by 0. In a narrow sense they are also rational since in the absence of sentiment traders, the zero steady state would be the rational solution.

Comparing those with rational agents, the dynamics are less stable with rational agents than with fundamentalists instead. This can be seen in Figure 5: a lower intensity of choice is required to offset the cyclic behavior induced by biased traders. This suggests that the hypothesis of an amplifying moment of rational agents is true as well: rational agents anticipate the behavior of boundedly rational agents and their resulting trading behavior induces a further moment of destabilization. This result is intuitive since fundamentalists will always believe that the future price equals its fundamental value. Their belief will hence will not be affected by the fact that that (other) boundedly rational traders have stronger beliefs or belief-switching occurs faster. This is in particular important for values of δ (here between 1.2 and 1.6) or β where rational agents already take the behavior of sentiment traders into account and hereby amplify their impact on the market.

4.4 The two-trader type model

Let us now turn to the 2-trader type model with rational agents vs. trend followers. In this model, there are only two types of agents, i.e. fully rational and trend chasing

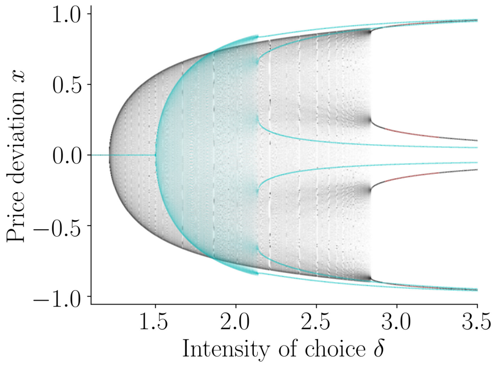

Figure 5: Bifurcations w.r.t. δ. In cyan the same simulation with fundamentalists instead of rational agents.

traders. The beliefs of trend chasers is given by

$$x_{t+1}^{e} = \gamma x_{t-1},$$

where γ denotes the degree of trend extrapolation. Note that, since the past prediction is part of the profit equation, the numerical method is now a mapping from $\mathbb{R}^3 \rightarrow \mathbb{R}$ which has consequences for the calculation speed.[5]

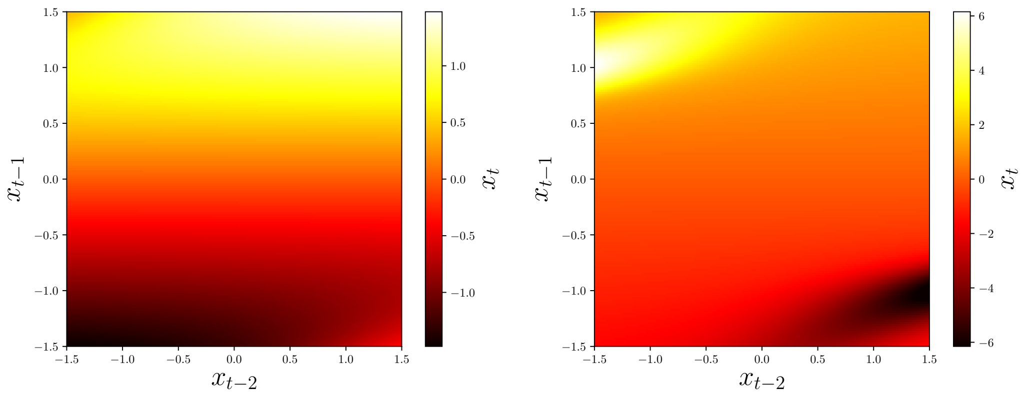

Figure 6: Heatmap of $g_{75}(x_{t-1}, x_{t-2}, 0)$ for $\gamma = 0.2$ (l) and $\gamma = 0.25$ (r). For the lower gamma, the zero steady state is still locally stable. For $\gamma = 0.25$ trajectories diverge slowly outwards. Note the different scales.

The actual dynamics are considerably simple. Given the parameters in Table 1 for γ smaller than approximately 1.24, the zero steady state is unique and stable. As γ increases further, the zero steady state becomes unstable and the system's dynamics

[5]For $\gamma \neq 0$ the function in (5) is defined on 3-D space, i.e. $g(x_{t-1}, x_{t-2}, x_{t-3})$ due to an extra time lag in the fractions $n_{i,t}$.

explode. This phenomena can be described by a so-called *hard bifurcation*.[6] The underlying forces are represented in Figures 6 and 7. Note that the functions represented here are actually two dimensional slices of the 3-dimensional function g. The fraction of (boundedly) rational agents is, after transition, constant and equal 0.5 for all values of γ. This is explained by the fact that in the zero steady state, the expectations of both types of agents are perfectly correct and switching probabilities are equal. When the system explodes, initialized by a positive price the trajectory is dominated by the beliefs of the trend chasers and, since agents that are forming rational expectations are always right, payoffs again are equal, leading to equal fractions of agent types.

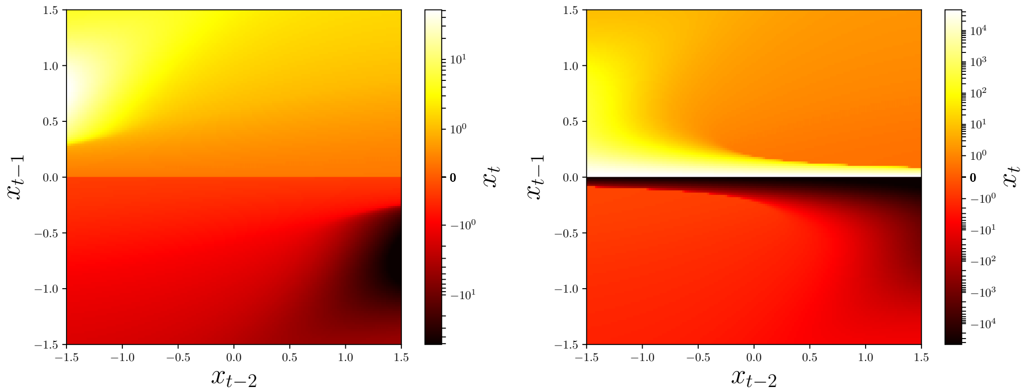

Figure 7: Heatmap of $g_{75}(x_{t-1}, x_{t-2}, 0)$ for $\gamma = 0.32$ (l) and $\gamma = 0.4$ (r). Trajectories lead from the center to the periphery. This effect amplifies with higher γ. Note the different scales.

This setup is difficult to asses and compare to the original results of Brock and Hommes (1998) and does not appear to link to their Lemmas 2 – 4, where the dynamics with respect to γ are related to values of $2R$ or R^2 respectively. While in their setup chaotic dynamics could be identified, this is not the case for the model described here.[7] It however strikes intuitive that, as soon as the degree of trend extrapolation reaches a certain threshold (which is well in line with their analytical results), rational agents will start to follow the trend chasers beliefs. This then further amplifies their beliefs, without any remaining force to revert the trajectory back to the steady state.

4.5 Risk adjusted fitness measure

As pointed out above, the fact that boundedly rational agents constitute a notable fraction of agents is due to the fact that in certain phases of the speculative cycle, these agents make higher profits than rational agents. This in turn is true because profits, and with that payoffs and feedback, are not proportional to agents' forecasting errors.

[6] To allow for comparison with the original results of Brock and Hommes (1998) I also conducted this simulation with their calibration of $R = 1.1$. The hard bifurcation is then, taking potential measurement errors into account, at $\gamma \approx 1.363$.

[7] Note that proving chaos is in general nontrivial. Likewise it is hard to show the absence of chaos for all variation of parameter values, in particular since simple results like *period-3 implied chaos* in general do not hold for the multidimensional case.

Again, as empathised in Brock and Hommes (1998), this can be attributed to the fact that the profit equation in (2) is correctly specified, but does not take investment risk into account. In particular, the payoff function considered before does adjust for the variance of asset prices. For this reason payoffs that are proportional to intra-period forecasting errors are another natural candidate for this analysis.

Hommes (2013, p.166 f.) shows that the payoff function then takes the form[8]

$$\pi_{i,t} = -(x_t - x^e_{i,t})^2 - \mathbb{1}_{RE}\kappa. \tag{7}$$

This rather reads as a punishment for forecasting errors than a payoff. It is immediately clear that the payoff for rational agents is *always* $-\kappa$, while payoffs for sentiment traders are $-(x_t \pm \beta)^2$. Then it is apparent that this system may have three steady states, each of them being associated to the dominance of one agent type. Figure 8 shows the associated bifurcation diagrams where steady states exchange stability at a Pitchfork bifurcation. Here the steady state where pessimistic traders dominate is ruled out because the simulation is initialized with a positive price. Note that for any $\kappa > 0$ a further increase of κ has the same effect on rational agents as an increase in δ since the fraction of rational agents evolves proportional to $e^{-\delta\kappa}$, which is relevant in particular since the fundamental steady state is stable and unique for all $\delta > 4.1$. This implies that, given a high intensity of choice, the costs of being rational have to be relatively small to shift the system back to a non-fundamental steady state.

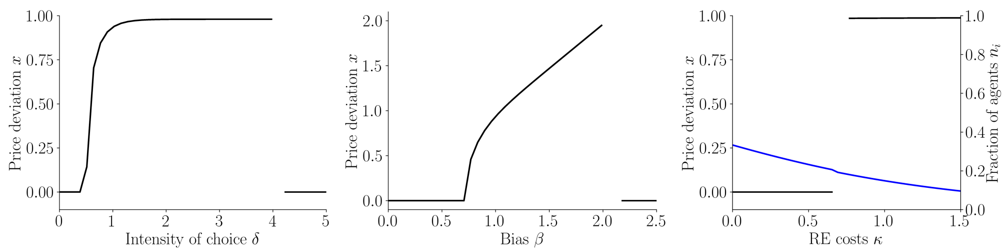

Figure 8: Bifurcation diagrams for δ (left), β (center) and κ (right). For the first two parameters the steady state increases with the parameter but falls back after a certain threshold. The last Figure depicts the case where $\delta = 5$. The blue line shows the fraction of rational agents.

These results again confirm the hypothesis that rational agents are stabilizing, in particular in favor of the fix-point argument. Note that we can not observe endogenous fluctuations in any of the simulations. Rather, a fix point is identified which is then either in accordance with the belief of one of the sentiment traders, or – if intensity of

[8]This takes into account the assumption of mean-variance utility and adjusts for the potential risk-free profits. In fact the full payoff function is then given by

$$\pi_{i,t} = -\frac{1}{2a\sigma^2}(y_t - y^e_t + \epsilon_{y,t})^2,$$

where a stands for the agents' risk-aversion and $\epsilon_{y,t}$ the stochastic fluctuations in prices with standard deviation σ. Since in 3 this term will be multiplied by δ for which no empirical counterpart is available, a precise adjustment for a and σ^2 would not provide further insight. Since I am focussing on the endogenous fluctuations in this work, the noise term can also be omitted.

choice or bias is sufficiently strong – returns to the fundamental value where sentiment traders' beliefs chancel each other out. The results for the simulations with increasing κ show clearly that a decreasing fraction of rational agents then leaves the market to either of the sentiment traders.

5 Conclusion

This work studies the dynamics of a simple financial market that is characterized by the coexistence of perfectly rational agents and sentiment traders, while the total number of each type varies proportional to their market performance. To find a solution to such model, one of my central contributions is to make use of iterative methods to find the rational expectations path.

The primary finding is that rational agents are prone to adapt beliefs of boundedly rational agents, which might amplify endogenous trading dynamics. This result is mainly driven by the self-fulfilling nature of asset price expectations, and amplified if fitness measures do not account for risk. This lack of stabilization by perfectly rational traders stems for the fact that they anticipate the behavior of boundedly rational agents and use this information in their trading decision.

Secondly, the numerical evidence sheds doubt on the proposition that boundedly rational agents are driven out of the market in the long run. Although with certain limitations, this is due to the strong feedback of expectations on prices and the fact that net profits are not proportional to forecasting errors. Depending on the magnitude of individual beliefs, speculative dynamics can emerge and coordination of rational and boundedly rational traders can become complicated, however not chaotic.

Further, in most setups the presence of rational agents does indeed tend to stabilize the market. This is in particular true if agents' fitness measure accounts for taken risk. However, the stable price might not necessarily reflect the economic fundamentals. Decreasing the amount of rational agents tends to destabilize the market, a result which again depends on the strength of sentiment beliefs. When beliefs are moderate, decreasing costs and increasing the degree of rationality in the market might in fact facilitate coordination and stabilize prices. If individual biases are strong the net-stabilizing effect might however be negligible.

In conclusion, my findings here support the hypothesis that rational agents tend to *stabilize* the market, but sentiment traders are not in general driven out of the market and can have considerable impact on prices. Given payoffs that do not account for risk correctly, the presence of boundedly rational agents also might induce endogenous oscillations that are further amplified by rational agents. Fully rational agents are then "riding the wave" and behave *as if* they are boundedly rational. These results shed further doubt on the propositions that financial markets are stable due to Rational Expectations Hypothesis.

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Appendix A The rational expectations path and the backwards consistent solution

To illustrate, let us take the most basic example of a forward looking dynamic system of the form

$$y_t = \alpha E_t y_{t+1}, \qquad |\alpha| < 1.$$

If we iterate this forward to $E_t y_{t+1} = \alpha E_t y_{t+2}$ and repeat this step infinitely many times the rational expectations solution is

$$y_t = \lim_{s\to\infty} \alpha^s E_t y_{t+s} = \begin{cases} 0 & \forall \lim_{s\to\infty} E_t y_{t+s} \in \mathbb{R} \\ \{\} & \text{if } —\lim_{s\to\infty} E_t y_{t+s}| = \infty. \end{cases}$$

This result is intuitive since $\lim_{s\to\infty} \alpha^s = 0$ for all $|\alpha| < 1$. If then the absolute value of $\lim_{s\to\infty} E_t y_{t+s}$ is infinite, y_t equals the product of zero and infinity which has no solution. But the above implies that any expectation $E_t y_{t+k}$ given $k > 0$ can either be 0 or has no solution, but an infinite solution is ruled out. It then follows that the Rational expectations solution, if it exists, is

$$y_t = 0 \; \forall t.$$

This reveals a common misconception. The Transversallity condition

$$\lim_{s\to\infty} E_t y_s = 0$$

guarantees a solution, but does not directly imply stationarity. This example can be generalized to the multidimensional case and has been treaded rigorously in the literature, see f.i. Blanchard and Kahn (1980) for the conditions on existence of a solution expressed in terms of the eigenvalues of the respective system.

The general argument for the PFP implies $E_t y_{t+1} = y_{t+1}$ and that at time $t-1$ the value of y_t must have been known in order to be able solve for y_{t-1}. By assumption, this value must have also been correct since agents have perfect foresight. It follows directly that it can be solved for y_t by just iterating the law of motion back one period. So if $y_t = \alpha y_{t+1}$, then it must also hold that $y_t = \alpha^{-1} y_{t-1}$ because y_{t-1} has already been chosen in the anticipation of y_t.

This conclusion however is false, which can be illustrated by looking at the stability characteristics of both systems under $|\alpha| < 1$. As shown above the RE system is stable and jumps back to zero from every point in $\mathbb{R}$. The perfect foresight path diverges unless the initial value of y_{t-1} lies on the REP i.e. is equal to zero. The system diverges for any set of initial values that does not lie on the RE path. Divergence then implies that the initial value for y_{t-1} must have already been off the REP, which is a direct contradiction to the conjecture that every perfect foresight solution also satisfies the condition of rationality.

Also, if agents posses complete knowledge of the system's LOM and the states that are relevant for this LOM, there is no reason to impose that further past information should be necessary to solve the expectations problem (such as y_{t-1} in the case of the example here).

Appendix B A note on convergence

While for the examples outlined here my algorithm (almost)[9] always converges, convergence is not guaranteed for any *finite* grid $\mathbf{x}$. If we continue Figure 1 with higher values of $\delta > 6$, even after many iterations the solution generally does not satisfy the convergence criterion. This problem is caused by the discontinuity of g for $\delta \to \infty$. To

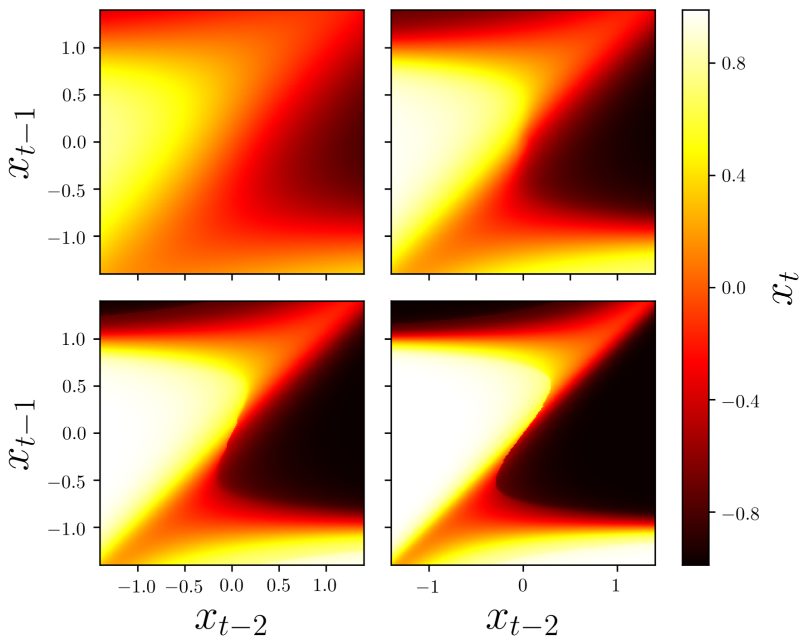

Figure B.9: Illustration of the numerical function g for $\delta = 1.1$ (ul), $\delta = 2.5$ (ur), $\delta = 4$ (ll) and $\delta = 7$ (lr). When δ increases the function becomes steeper and at the diagonal a small change in x_{t-1}, x_{t-2} leads to a larger change in $x_t = g(x_{t-1}, x_{t-2})$.

understand, imagining the numerical function g on the grid $(\mathbf{x}, \mathbf{x}')$. In my solution algorithm it is necessary to evaluate $g_k(g_k(\cdot), \mathbf{x})$, i.e. to have a real valued input into function defined on a discrete space, which numerically necessarily involves an interpolation. Let us assume we want to evaluate g_k at a point $z \in \mathbb{R}$ for which no $g_k(z, \cdot)$ exists, let us call X_k the nearest smaller grid point $X_j < z$ for which $g_k(X_j, \cdot)$ exists, and $X_{j+1} > g_k(\cdot)$ the nearest higher point respectively. Accuracy of the interpolation then naturally decreases with $\Delta G = |g_k(X_j, \cdot) - g_k(X_{j+1}, \cdot)|$.

[9] In Figure 2 the region around $\beta \in (1.7, 1.8)$ does not converge. The same occurs for δ between approximately 3 and 3.4 in Figure 5. I use red to mark the values where the procedure did not converge.

Figure B.9 shows g_{500} for increasing values of δ. While for $\delta = 1.1$ the function is very smooth, it becomes steeper when δ gets larger. For $\delta = 7$ in the last diagram the function is very steep with ΔG being particularly high in the center and decreasing in the periphery. This region close to the center and along the diagonal is not captured well by the algorithli since small deviations in the respective $g_n(\cdot)$-values lead to large differences in $g_{n+1}(\cdot)$-values in the next iteration. This problem can be partially mitigated by increasing the size of the grid and inserting a new node X_{new} such that $X_k < X_{\text{new}} < X_{k+1}$. Now $g(z, \cdot)$ will be evaluated as the interpolation between X_k and X_{new} (assuming $z < X_{\text{new}}$) which probably exhibits a lower ΔG (though not with certainty).

In fact, for every finite grid there will always be a combination of parameters for which there exists a X_k and X_{k+1} for which ΔG is large. This problem can generally be tackled quite efficiently by implementing an Endogenous Grid Method that allocates relatively more grid points to the critical region. This however is not necessary for the example here. Since we know that the center will reflect the trajectory back to the periphery, and by noting that the periphery also redirects to the periphery we can conjecture that the center has little effect on the simulation of the time series. For this reason, if necessary, I take the 500th iteration of g, $g_{500}(\cdot)$, and use it to simulate the time series.[10] The result confirms that the conjecture was correct.

[10] In fact in most cases g_{25} is already sufficient and accurate.